## THE MAY MEETING IN NEW YORK

The two hundred forty-ninth regular meeting of the Society was held at Columbia University, on Saturday, May 1, 1926, extending through the usual morning and afternoon sessions. The attendance included the following sixty-four members:

W. L. Ayres, C. R. Ballantine, J. P. Ballantine, A. A. Bennett, Eisenhart, Fiske, Fite, Fort, Frink, Gafafer, Garretson, Gill, R. L. Green, Gronwall, Hansell, Hedlund, Hille, Huntington, Joffe, Kasner, Kormes, Lamond, Lamson, Langford, Langman, Littauer, Maria, Meyer, H. H. Mitchell, L. T. Moore, T. W. Moore, Murnaghan, Norwood, B. C. Patterson, Pfeiffer, Pierpont, R. G. Putnam, Rainich, Raudenbush, Raynor, Reddick, Ritt, Schwartz, Seely, Siceloff, M. H. Stone, J. H. Taylor, Teach, J. M. Thomas, Tracey, Veblen, H. E. Webb, Weida, Weiss, Pell-Wheeler, Whittemore, W. A. Wilson.

There was no meeting of the Council or of the Trustees.

Professor E. W. Brown presided at the morning session, relieved by Professor Oswald Veblen; in the afternoon Professor L. P. Eisenhart presided, relieved by Professor J. F. Ritt.

At the request of the Program Committee, Professor H. H. Mitchell delivered an address, at the beginning of the afternoon session, entitled *Asymptotic laws in the theory of numbers*. Titles and abstracts of the other papers read at the meeting follow below. The papers of Adams, Carman, Cleveland, Emch, Foster, Gehman, Georges, Glenn, Graves, Gronwall (second paper), Hille, Hollcroft, Kellogg, Levy, Lubben, Maria, Morley, Robertson, Robinson, Tamarkin, Tamarkin and Langer, Vandiver, and Whyburn were read by title. Mr. Cleveland and Mr. Whyburn were introduced by Professor R. L. Moore, Mr. Georges by Professor Dickson, Professor LaMer by Dr. Gronwall, and Professor Starke by Professor Ritt.

1. Mr. T. W. Moore: *Notes on the rational plane cubic curve.*

This paper appeared in full in the May-June issue of this BULLETIN.

2. Professor T. R. Hollcroft: *On harmonic cubics.*

It has long been known that the Hessian of the Hessian of a harmonic cubic is the original cubic. The process of obtaining the Hessian of a canonical cubic may be considered merely a repeated transformation of the parameter. Define the process of obtaining the Cayleyan of a canonical cubic successively as a similar repeated transformation of the parameter, the curve being dualized each time the transformation is applied. Then it is also true that the Cayleyan of the Cayleyan of a harmonic cubic is the cubic itself. Moreover, the Cayleyan of any harmonic cubic is harmonic and when the cubic is real, the Cayleyan has the same parameter as the cubic. The condition that the parameter of the Cayleyan of the Hessian equal the parameter of the Hessian of the dualized Cayleyan of a given cubic is satisfied by all the fourteen special cubics of the syzygetic pencil, six harmonic, four equianharmonic, and four degenerate, and by no others.

3. Professor James Pierpont: *Note on the horospheres.*

This paper will appear in an early issue of this BULLETIN.

4. Professor Frank Morley: *Note on differential inversive geometry.*

In the memoirs of Mullins,* Liebmann,† and Kubota‡ the differential inversive geometry of a planar curve is firmly established. The following way of approach seems desirable. Denote the Schwarzian derivatives of $x$ as to $y$ by $\{x,y\}$. We have, for any number of connected variables Cayley's cyclic formula (say $C_n$), $\{x,y\}(dy)^2+\{y,z\}(dz)^2+\cdots+\{l,x\}(dx)^2=0$. Let the curve in question be given by a self-conjugate equation between $x$ and its conjugate $\bar{x}$. Then by $C_2$, $\{x,\bar{x}\}(d\bar{x})^2+\{\bar{x},x\}(dx)^2=0$; so that $\{x,\}\bar{x}(d\bar{x})^2=2i(dt)^2$, where $t$ is real. Let us next apply a homography $x=(ay+b)/(cy+d)$, and use $C_4$ for the cycle $x,\bar{x},\bar{y},y$. Since $\{y,x\}=0$ and $\{\bar{x},\bar{y}\}=0$, we have $\{x,\bar{x}\}(d\bar{x})^2+\{\bar{y},y\}(dy)^2=0$, or $\{x,\bar{x}\}(d\bar{x})^2=\{y,\bar{y}\}(dy)^2$. Hence $t$ is an invariant under homographies, and is the proper real parameter. Using $C_3$, we find $\{x,t\}(dt)^2-\{\bar{x},t\}(dt)^2=\{x,\bar{x}\}(dx)^2=2i(dt)^2$; so that $\{x,t\}=i+I$, $I$ being the fundamental differential invariant.

5. Mr. B. C. Patterson: *A formula for the inversive curvature.*

If a curve is given as a complex function of a real parameter, let the Schwarzian derivative of this function be separated into its real and imaginary parts, $A$ and $iB$. The inversive curvature is then given by the formula $I_5=(8AB^2-4BB''+5B'^2)/8B^3$, where the primes indicate differentiation with respect to the parameter.

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* *Differential Invariants under the Inversion Group*, Columbia dissertation, Lancaster, Pa., 1917.

† *Beiträge zur Inversionsgeometrie der Kurven*, MÜNCHENER BERICHTE, 1923.

‡ *Beiträge zur Inversionsgeometrie*, SCIENCE REPORTS, TÔHOKU UNIVERSITY, Mathematical Section, vol. 3 (1924), No. 1.

6. Mr. G. Y. Rainich: *Remarks on differential inversive geometry.*

(1) If we call inversive distance between two linear elements of the second order the square root of the sine of the angle (which may be imaginary) formed by the two circles determined by the elements, the inversive length of arc may be defined as the limit toward which tends the sum of inversive distances between consecutive elements on the arc when these distances tend towards zero. (2) It is known that using dual numbers the Laguerre inversions may be given in the form of fractional linear transformations. With the aid of this representation, two seminvariants of plane curves under the Laguerre group are obtained, by forming the Schwarzian derivative, and from these the differential and integral invariants are derived, in the same way as this is done for point inversions.

7. Mr. G. Y. Rainich: *A type of space-time.*

The expression in vector form is given for a Riemann tensor which allows a two-parameter group of rotations. This tensor depends on eight constants. The Bianchi equations are written out in vector form, and a type of space-time the Riemann tensor of which depends on seven constants is discussed, for which the radiation vector $q$ vanishes in regions of regularity, and, as a consequence, Maxwell's equations are satisfied.

8. Professor F. D. Murnaghan: *A contribution to Maxwell's theory of dielectric absorption.*

According to Maxwell, the residual or "anomalous" currents which follow the charge or discharge of a dielectric are due to a lack of homogeneity of the dielectric, and he indicated a theory of a layer dielectric but did not carry out the solution. The current is given by a linear integral equation of Volterra's type whose solution consists of a finite number of exponential terms. The complete solution for continuous and alternating E.M.F.'s follows readily. The solution furnishes a proof of the important Boltzmann-Hopkinson empirical "principle of superposition."

9. Mr. W. L. Ayres: *Continuous curves of a certain type.*

S. Mazurkiewicz,* R. L. Wilder,† and others have considered continuous curves which contain no simple closed curve. The present paper considers continuous curves containing only a finite number of simple closed curves, and it is found that these curves possess many properties similar to those proved for the above type. Among other properties given, it is shown that for this type every connected subset is arc-wise connected, every closed connected subset is a continuous curve, and every boundary point of an $S$-domain of such a curve is accessible. Some of the properties

* *Un théorème sur les lignes de Jordan*, FUNDAMENTA MATHEMATICAE, vol. 2 (1921), pp. 119–30.

† *Concerning continuous curves*, FUNDAMENTA MATHEMATICAE, vol. 7 (1925), pp. 341–77.

are possessed also by the more general type such that, given any positive number $\epsilon$, the continuous curve contains only a finite number of simple closed curves of diameter greater than $\epsilon$.

10. Mr. W. L. Ayres: *On continuous* (1,1) *correspondences between continuous curves.*

The author considers the following problem. Suppose $M$ and $M'$ are continuous curves and there exists a (1,1) correspondence between $K$ and $K'$, subsets of $M$ and $M'$ respectively. Under what conditions can this correspondence be extended to a (1,1) continuous correspondence between the curves themselves? In this paper the conditions are given for the cases where the continuous curves contain no simple closed curve and where they contain just one simple closed curve. Branch points are defined, and it is proved that in the two cases under consideration the set of branch points is countable. The conditions given are conditions on the branch points of the two curves and on $K$ and $K'$. For the case where $K$ and $K'$ are vacuous, this gives a set of conditions on the branch points under which two continuous curves can be put into (1,1) continuous correspondence. The conditions given are independently necessary.

11. Mr. Mark Kormes: *On arithmetic summation of point sets.*

Arithmetic summation of point sets is defined as summation of the coordinates of their points. The following theorems are shown to hold: (1) the arithmetic sum of two sets having positive measure contains at least one interval; (2) the arithmetic sum of a single set of positive measure contains at least one interval; (3) the preceding theorems are true if we replace the sets having positive measure by sets the complements of which with respect to an interval are of the first category.

12. Dr. C. H. Langford: *On two new quantifiers for general propositions.*

General propositions are commonly constructed in terms of the two applicatives "some" and "every." These applicatives occur in functions of the form $(\exists x) \cdot \phi x$ and $(x) \cdot \phi x$. There are four quantitative functions of a property $\phi\hat{x}$, namely, every $x$ is such that $\phi x$, some $x$ is such that $\phi x$ there is no $x$ such that $\phi x$, and not every $x$ is such that $\phi x$. "No $x$ is such that $\phi x$" is denoted by $[x] \cdot \phi x$, and "not every $x$ is such that $\phi x$" by $\{x\} \cdot \phi x$. In connection with multiply-quantified functions both of these latter applicatives have formal properties which are in some respects simpler than the properties of "some" and "every." $[x] \cdot \phi x$ generalizes Sheffer's function $p$ "stroke" $q$ when interpreted "$p$ and $q$ are both false" and $\{x\} \cdot \phi x$ generalizes $p$ "stroke" $q$ when interpreted "$p$ is false or $q$ is false."

13. Dr. M. G. Carman: *Expansion problems in connection with homogeneous linear q-difference equations.*

Expansions of the Neumann and Gegenbauer types are obtained for analytic functions in terms of solutions of the general homogeneous linear $q$-difference equation with analytic coefficients. Expansions analogous to ascending power series, descending power series, and Laurent series are derived. The question of expanding functions of several variables is considered. This paper will appear in an early issue of the TRANSACTIONS OF THIS SOCIETY.

14. Dr. Rachel B. Adams: *The solutions of Fredholm's homogeneous integral equation as functionals of the kernel.*

This paper deals with the Fredholm homogeneous integral equation of the second kind in which the kernel $K(x,y)$ is real and continuous in the region $S$ considered. We first show that the characteristic numbers vary continuously with the kernel. Let, then, $\lambda_0$ of multiplicity $n$ and $\bar{\lambda}_i$ $(i=1, 2, \cdots, n)$ be corresponding characteristic numbers for $K(x,y)$ and $\bar{K}(x,y)$ respectively. If the indices of $\lambda_0$ and of the $\bar{\lambda}_i$ are equal [unequal] to the corresponding multiplicities, it is shown that given an arbitrarily small positive quantity $\epsilon$, a positive number $\delta$ can be found such that if $|K(x,y) - \bar{K}(x,y)|$ is $<\delta$ throughout $S$, the difference between each of the $n$ normalized fundamental solutions (Lalesco) [each of the normalized fundamental solutions] of either kernel and a suitable linear combination of the $n$ normalized fundamental solutions [functions] of the other is in absolute value less than $\epsilon$. It is of interest to note that we thus obtain for both cases approximate solutions of the given integral equation. A method is developed for determining the closeness of this approximation.

15. Professor J. Tamarkin: *On Laplace's integral equation.*

This paper deals with the solution $F(x)$ of the equation

$$\int_0^\infty e^{-zx} F(\xi)d\xi = f(z),$$

as a function of the complex variable $x$. Certain properties of reciprocity between $f(z)$ and $F(x)$ are established.

16. Professor J. Tamarkin: *On Volterra's integro-functional equation.*

This paper is devoted to the discussion of the theorems of existence and uniqueness of solutions of the integro-functional equation

$$u(x) = f(x) + s(x)u[\theta_1(x)] + \int_{-\infty}^{x} K(x,\xi)u[\theta_2(\xi)]d\xi,$$

where $u(x)$ is the unknown function and all other functions are given.

17. Professors J. Tamarkin and R. E. Langer: *A notion of uniform integrability.*

This paper appears in full in the present issue of this BULLETIN.

18. Professor H. S. Vandiver: *On the least multiple of an integer expressible as a definite quadratic form.*

This paper contains a proof of the following theorem: If $r^2+a\equiv 0$

(mod $n$), where $a>0$, $a$ is not a square and prime to $n>1$, then there exists at least one set $(x,y)$ such that $x^2+ay^2=kn$, $k\leqq 2\sqrt{a}/3$, and $ry\equiv\pm x$ (mod $n$), $x$ and $y$ being integers. The proof is easily obtained from the theory of definite quadratic forms. The result is required in connection with one of the author's methods of factoring large integers.

19. Professor H. S. Vandiver: *Summary of results and proofs concerning Fermat's last theorem.* Second note.

In this note, among other results, the author indicates the proofs of the following theorems: If $p$ is an odd prime, and $u^p+v^p+w^p=0$ is satisfied in non-zero integers in the field $\Omega(\alpha+\alpha^{-1})$, $\alpha=e^{2i\pi/p}$, then the class number of the field $\Omega(\alpha)$ is divisible by $p^2$. Under the assumptions (1) none of the Bernoulli numbers $B_k$, $k=(sp+1)/2$, $B_1=1/6$, $B_2=1/30$, etc. ($s=1, 3, \cdots, p-4$) are divisible by $p^2$, for $p$ an odd prime, and (2) the second factor of the class number of the field $\Omega(\alpha)$ is prime to $p$, the equation $x^p+y^p+z^p=0$ is not solvable in rational integers, none zero.

20. Dr. H. M. Gehman (National Research Fellow): *On extending a continuous* (1,1) *correspondence of two closed point sets.*

In this paper it is proved that if $M$ and $M'$ are closed point sets lying in euclidean $n$-spaces $L$ and $L'$ respectively, and if $S$ and $S'$ are euclidean $(n+1)$-spaces containing $L$ and $L'$ respectively, and if there exists a continuous (1,1) correspondence $T$, such that $T(M)=M'$, then there exists a continuous (1,1) correspondence $U$, such that $U(S)=S'$, and such that for any point $P$ of $M$, $U(P)=T(P)$. This theorem has already been proved for the case $n=1$ by R. L. Moore (AMERICAN JOURNAL, vol. 48 (1926), p. 67).

21. Dr. R. G. Lubben: *Concerning fixed points in an acyclic continuum undergoing a continuous transformation.*

If $K$ is a bounded continuum such that no sub-continuum of $K$ other than a point is its own prime part, not more than one sub-continuum of $K$ is irreducible between any two given points of $K$, and $T$ is a continuous one-to-one transformation, with a single-valued inverse, of $K$ into itself, then $T$ leaves fixed at least one point of $K$.

22. Mr. G. T. Whyburn: *Concerning domains and their boundaries.*

In this paper the following theorems are established. (I) In order that a continuous curve $M$ should be the boundary of a connected domain it is necessary and sufficient that if $J$ denotes any simple closed curve belonging to $M$, $I$ its interior, and $E$ its exterior, then (a) $M$ is a subset either of $J+I$ or of $J+E$, and (b) if $A$ and $B$ denote any two points of $J$, then $M-(A+B)$ is not connected. (II) If the boundary $M$ of a domain $D$ is a continuous curve, and if $K$ denotes the set of all the cut points of $M$, then

$D+K$ is uniformly connected im kleinen. (III) If the boundary $M$ of a simply connected bounded domain $D$ has only a finite number of complementary domains, then in order that $M$ should be a continuous curve it is necessary and sufficient that $M$ should contain a closed point set $K$ such that (a) $D+K$ is uniformly connected im kleinen, (b) every maximal connected subset of $K$ is a continuous curve, and (c) for every number $\epsilon>0$, there are only a finite number of maximal connected subsets of $K$ of diameter $>\epsilon$.

23. Mr. G. T. Whyburn: *Two-way continuous curves.*

A continuous curve $M$ will be said to be "two-way continuous" if and only if between every two points of $M$ there exists in $M$ at least two arcs neither of which is a subset of the other. The following theorems are established. (1) A continuous curve $M$ is two-way continuous if and only if every arc of $M$ contains a non-cut point of $M$. (2) A continuous curve $M$ is two-way continuous if and only if the set of all the cut points of $M$ contains no continuum. (3) The boundary of a complementary domain of a two-way continuous curve is itself two-way continuous. (4) The boundary $M$ of a complementary domain of a continuous curve is two-way continuous if and only if $M$ contains a point set $K$ such that (a) $D+K$ is uniformly connected im kleinen, and (b) every arc, if any, which $\overline{K}$ ($K$ plus its limit points) contains, contains a non-cut point of $M$. (5) If $M$ is any continuum whatever, and if $L$ denotes the set of all the cut points of $M$, then every continuum which $L$ contains is a continuous curve.

24. Mr. C. M. Cleveland: *Conditions under which two points of a continuous curve are separated by a simple closed curve lying therein.*

The author establishes the following theorem. If $A$ and $B$ are two distinct points of a continuous curve $M$ such that any simple continuous arc from $A$ to $B$ contains at least one point of $M$ distinct from $A$ and from $B$, then there exists a simple closed curve lying wholly in $M$ and separating $A$ from $B$.

25. Dr. L. M. Graves (National Research Fellow): *Implicit functions and differential equations in general analysis.*

The chief purpose of this paper is to discuss some special cases of the implicit function theorems obtained by Hildebrandt and Graves in a paper entitled *Implicit functions and their differentials in general analysis.* A number of general theorems are first given, which facilitate application of the general theory to special cases. An "extended" implicit function theorem is obtained by a new method from the original implicit function theorem. In the treatment of existence and imbedding theorems for ordinary "differential equations," continuity restrictions are removed and no mention is made of derivatives. The treatment seems to be the most general that is possible with use of a Lipschitz condition. As a special case, imbedding and existence theorems for solutions of equations in the form

$f(x, \int_a^x g(x, x', y(x'), c)dx', y(x), c) = 0$ are treated in detail. The functions $f$, $g$, and $y$ are supposed to be bounded and measurable in the single real variable $x$. Finally, some special properties of solutions of linear "differential" equations are discussed.

26. Dr. H. P. Robertson (National Research Fellow): *Manifolds with non-Riemannian line element which admit geodesic coordinates.*

It is asked under what conditions an $n$-dimensional manifold, whose metrical properties are determined by an invariant quartic differential form, admits geodesic coordinates. This requires that the $(1/24)n(n+1)(n+2)(n+3)$ coefficients of the form are subject to $(1/24)n^2(n^2-1)(n+6)$ first order differential equations linear and homogeneous in the derivatives. These equations have been solved for the case $n=2$, the case treated by the classical differential geometry, and show that the only surfaces for which the quartic does not degenerate into the square of a quadratic are those in which the line element may be reduced on transformation to one with constant coefficients. The analogous solutions satisfy the equations for any value of $n$, but for $n>2$ the above method becomes too cumbersome, so the search for other possible solutions is being continued by the methods of the absolute differential calculus. Of particular interest from the physical standpoint is the case $n=4$, as its solution will give the only possible "physical universes" which are in harmony with the postulates of the theory of relativity concerning the nature of force.

27. Dr. Harry Levy (National Research Fellow): *The canonical form of the $ds^2$ of spaces for which the five-index symbols of Riemann vanish.*

The author shows that the $ds^2$ of spaces for which the five-index symbols of Riemann vanish is expressible as the sum of forms, $ds^2=\sum d\sigma_\alpha^2$, where $d\sigma_\alpha^2$ is a form of constant curvature and such that the variables of any one $d\sigma_\alpha^2$ appear only in that $d\sigma_\alpha^2$.

28. Dr. Harry Levy: *Some properties of spaces for which the five-index symbols of Riemann vanish.*

Application is made of the results of the preceding paper to obtain several properties of those spaces for which the five-index symbols of Riemann vanish. In particular, the linear element of the most general space which can be mapped geodesically upon a given space of the above type is found to be $ds^2=\sum_\alpha \rho_\alpha d\sigma_\alpha^2$, where the $\rho_\alpha$ are constants, a priori arbitrary, and where $ds^2=\sum_i d\sigma_\alpha^2$ is the $ds^2$ of the given space. Other properties are obtained; also conformal maps between two spaces of this type are studied.

29. Dr. Harry Levy: *Congruences of curves in the geometry of paths.*

The author considers, in the geometry of paths, $n$ independent congruences of curves, and, upon them as basis, he develops a theory which is quite analogous to the theory of the orthogonal ennuple in a Riemann space as developed by Ricci, and which includes this theory due to Ricci as a special case. Certain functions of the ennuple are obtained and then found to be invariant under transformations of coordinates and to have properties which mark them as being the analogues of Ricci's coefficients of rotation. Properties of congruences of curves or of the space itself are expressed in terms of these invariants. In addition *projective* invariants are discussed, and in terms of them projective properties of the space are expressed.

30. Professor M. C. Foster: *Ruled surfaces referred to the trihedral of a directrix.*

The directrix is taken as any curved line $C$ upon the ruled surface. The displacements of any point on the ruling through a point $M$ of $C$ are considered relative to the moving trihedral of $C$ at $M$. Many of the well known theorems concerning ruled surfaces are proved by this method. The paper contains certain theorems relative to ruled surfaces for which the line of striction is an asymptotic line, and to the locus of the center of geodesic curvature of an orthogonal trajectory of the rulings.

31. Dr. A. J. Maria (National Research Fellow): *Stieltjes derivatives.*

In this paper we consider the derivative of a completely additive function of sets $f(e)$ with respect to $g(e)$ ($g(e)$ a completely additive function of sets and not necessarily the measure function). Theorems are obtained concerning the decomposition of $f(e)$ into a singular function and a function which has $g(e)$ as a basis function.

32. Professor Arnold Emch: *On the discriminant of ternary forms and a certain class of surfaces.*

In a paper entitled *On the Weddle surface and some analogous loci*,* the author studied the class of surfaces and curves defined as the loci of vertices of general cones passing through certain definite numbers of generic fixed points in space. This investigation may be extended to such systems of cones with nodal generatrices. For this purpose, it is obviously necessary to make use of the properties of the discriminant of a ternary form, to which the intersection of a cone with a generic plane reduces. The principle result is contained in the following theorem: The locus of vertices of $n$-ic cones through $(1/2)n(n+3)$ generic points in space is a surface of order

* TRANSACTIONS OF THIS SOCIETY, vol. 27 (1925), pp. 270–78.

$(1/2)(n-1)^2n^2(n+3)$, which has each of these points as a multiple point of order $3n(n-1)^2$, and each join of two of these points as a multiple line of order $3(n-1)^2$. As an example, the locus of vertices of cubic cones is investigated in detail.

33. Mr. J. S. Georges: *On division algebras corresponding to abelian groups with n abelian generators.*

This paper is a generalization of the results obtained by L. E. Dickson on an independent set of associativity conditions for division algebras corresponding to an abelian group with two independent generators, and consists of a complete determination of an independent set of associativity conditions, between the parameters, for division algebras corresponding to abelian groups having a set of $n$ independent generators.

34. Professor Einar Hille: *Applications of a Tauberian theorem to the convergence theory of orthogonal series.* Preliminary report.

Littlewood has proved that a series is convergent to the sum $A$, if (*i*) it is summable Abel to $A$, and (*ii*) the terms are $O(1/n)$. As far as the author knows, this theorem has not previously been applied to the convergence theory of orthogonal series other than Fourier series. In applying the theorem to other series we have to prove summability Abel of $\sum f_n u_n(x)$ for certain functions $f(x)$, and to give examples of functions for which $f_n u_n(x) = O(1/n)$. Theorems on summability Abel are known in a few important cases, and can be proved for more extensive classes of expansions. One can frequently prove that the terms are $O(1/n)$ when $f(x)$ is a saltus function; it follows that certain classes of functions which can be approximated by saltus functions have the same property. If $f(x)$ is of bounded variation and there exists a $p(x)$ such that $c_n = \int_a^b p(x)u_n(x)df(x)$ can be expressed linearly in terms of the $f_n$, then it is sometimes possible to obtain such an estimate of $c_n$ that the terms in the orthogonal series become $O(1/n)$. The author has tried out these ideas with a good deal of success on certain expansions in terms of polynomials (Hermite's, Laguerre's, and Legendre's) or boundary value functions of the Sturm-Liouville type.

35. Dr. Tobias Dantzig: *On systems of conics associated with point-to-point transformations of the plane.*

Let $T$ be a continuous point-to-point transformation of the plane in itself, $X$ and $Y$ two corresponding points. The line elements through $X$ and $Y$ are in homographic correspondence, and the intersections of corresponding rays determine a conic $(C)$ through $X$ and $Y$. The aggregate of the conics $(C)$ form a two-parameter family $(S)$, the indicating system of the transformation $T$. This system was considered in an earlier paper by the author (AMERICAN JOURNAL, 1919). In the present paper the inverse problem is discussed: Given a two-parameter family of conics $(S)$, to determine the transformations $T$ which admit $S$ for an indicating system.

For the general case, the conditions for the existence of $T$ are determined in the form of partial differential equations connecting the canonical parameters of the conic $(C)$ with the two parameters of the system $(S)$. Results of geometric interest are obtained for certain special cases.

36. Mr. P. D. Schwartz: *On the classification of conics in hyperbolic geometry.*

In this paper a third degree equation is derived whose roots determine the poles of an ortho-polar triangle. By considering the different combinations of the roots of the cubic, and by taking first poles not on the absolute and second poles on the absolute, and by means of transformations, the conics in hyperbolic geometry are classified.

37. Professor O. E. Glenn: *On biternary transvection and reducing series with application to systems of invariants of connexes.* Preliminary report.

Although several mathematicians have proposed forms of generalization of Gordan's series, the theory, as discovered to date, leaves much to be desired. The present writer has identified a new algorithm in the symbolic theory which extends the most general form of the reducing series of Gordan to the case of biternary concomitants. This sequence, as applied to the problem of reduction of ternariants, approximates in utility to the binary series of Stroh, for which the claim was made that it would give all reductions. Applications in this paper in its present form include a new derivation of the known system of eight concomitants of $a_x a_u$ (Clebsch and Gordan) and the fundamental forms of $a_x^2 a_u$ of the lower degrees. The number of invariant formations for $a_x^m a_u^n$ increases with such rapidity with the orders $(m,n)$ that a theory of symbolic moduli is desirable and is being developed

38. Professor E. P. Starke: *Certain uniform functions of rational functions.*

The functions $\cos z$ and $\wp(z)$, being even, are uniform functions of $z^2$. This suggests examining various classes of meromorphic functions, to determine which ones among them have the property of being uniform functions of non-linear rational functions. In this paper, the examination is carried out for the periodic functions and for the Poincaré functions (with rational multiplication theorems). We show that periodic functions with the above property either are meromorphic functions of $\cos(\lambda z+\mu)$, or are rational functions of $\wp(z+\mu)$, or of $\wp^2(z+\mu)$ if $g_3=0$, or of $\wp'(z+\mu)$ or of $\wp^3[(z+\mu)$ if $g_2=0$. In regard to the Poincaré functions, we note that if in one such function, we replace $z$ by any integral power of $z$, we have another Poincaré function with the property we seek. All Poincaré functions which satisfy our requirements and cannot be obtained in the above manner are proved periodic. They are then linear functions of $\cos(\lambda z+\mu)$, or of $\wp(z+\mu)$, or of $\wp^2(z+\mu)$ if $g_3=0$, or of $\wp^3(z+\mu)$ or $\wp'^2[(z+\mu)$ if $g_2=0$, with restrictions on the values of $\mu$.

39. Dr. T. H. Gronwall and Professor V. K. LaMer: *Internal and free energy in their relation to the Debye-Hückel theory.*

It is shown that, contrary to previous statements in the literature, the validity of the Debye and Hückel derivation of the expression for the free energy from that of the internal energy is in no wise dependent on the assumption (which is contrary to experimental facts) that the dielectric constant is independent of the temperature. Conditions on the dielectric constant, sufficient for the validity of the theory, are given, and these conditions do not disagree with any of the experimental data. The remainder of the paper discusses certain thermodynamical errors which have appeared in the literature in this connection.

40. Dr. T. H. Gronwall: *On Gibbs' phenomenon.* Second paper.

This paper contains certain theorems supplementary to those given in the paper read at the February, 1925, meeting of the Society, and also the numerical evaluation of the value of the Cesàro summation order above which Gibbs' phenomenon ceases to exist.

41. Mr. L. B. Robinson: *On equations in mixed differences.*

The author considers the equation $u'(x)=\sum_{j=1}^{n}P_j(x)u(a_jx+\delta_j)+Q(x)$, with $|a_j|<1$, $P_j(x)$ and $Q(x)$ polynomials, $u(x)=c_0+(c_1/\mu)x+(c_2/2\mu^2)x^2+\cdots$, $|\mu|<|\delta_j|(1+|a_j|+|a_j^2|+\cdots)$. The $c$'s are given by solving the system $\mu_1=\lambda_{01}c_0+[1+\lambda_{11}]c^1+\lambda_{21}c_2+\cdots$, $\mu_2=\lambda_{02}c_0+\lambda_{12}c_1+[1+\lambda_{22}]c_2+\cdots$, $\mu_3=\lambda_{03}c_0+\lambda_{13}c_1+\lambda_{23}c_2+\cdots$, $\cdots$. The above system has a converging determinant, and the $c$'s all satisfy the inequality $|c_i|<|M|$ $(i=1, 2, 3, \cdots)$, where $M$ is some finite constant. The element $c_0$ is arbitrary. So the solution converges. He also considers the case $u'(x)=\sum_{j=1}^{n}(P_j(x)/x^k)u(a_jx+\delta_j)+Q(x)/x^k$, $\mu|a_j|+|\delta_j|<\mu, |a_j|\mu/\delta_j<1$. In this case, the solution has a logarithmic singularity at the origin, and a pole of order $k-1$.

42. Professor O. D. Kellogg: *Regular and exceptional boundary points for the general Dirichlet problem.*

Green's function for the general open continuum is defined, and some of its properties are developed. Regular and exceptional boundary points are defined as those at which Green's function does or does not approach zero, respectively. Simple derivations are given of important properties of Wiener's sequence solution of the Dirichlet problem, and some theorems on capacity are proved.

R. G. D. RICHARDSON,
*Secretary.*

# A BOUNDARY VALUE PROBLEM IN THE CALCULUS OF VARIATIONS*

BY G. A. BLISS

1. *Introduction.* It is well known that the necessary condition of Jacobi for a problem with fixed end-points in the calculus of variations is closely related to a certain boundary value problem associated with Jacobi's differential equation.† If Jacobi's condition is satisfied then the smallest value $\lambda_1$ of the parameter $\lambda$ of the boundary value problem, for which that problem has a solution, must satisfy the inequality $\lambda_1 \geqq 0$, and conversely. In the following pages it is proposed to deduce a similar relationship for problems of the calculus of variations with variable end-points. The corresponding boundary value problem has end conditions of a more general type than those which arise when the end-points of the original calculus of variations problem are fixed. The existence of the smallest value $\lambda_1$ is established by methods of the calculus of variations, in particular by means of a theorem analogous to a well known theorem of Osgood.‡ It seems likely that a complete theory of self-adjoint boundary value problems for ordinary differential equations, with end conditions of a very general type, can be deduced from theorems already well known in the calculus of variations. So far as I know this has never been done, though many significant relationships have of course been pointed out.§

2. *The Calculus of Variations Problem and its Second Variation.* Let $C_1$ and $C_2$ be two arcs with the parametric equations

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* Presented to the Society, April 14, 1922.

† See, for example, Lovitt, *Linear Integral Equations*, p. 207.

‡ TRANSACTIONS OF THIS SOCIETY, vol. 2 (1901), p. 273.

§ See, for example, Richardson, MATHEMATISCHE ANNALEN, vol. 68 (1910), p. 279; Plancherel, BULLETIN DES SCIENCES MATHÉMATIQUES, vol. 47 (1923), p. 376.

$(C_i)$ $\qquad x = X_i(u), \quad y = Y_i(u), \qquad (i = 1, 2)$

and let $E$ be an arc

$(E)$ $\qquad y = y(x) \qquad (x_1 \leqq x \leqq x_2)$

intersecting them, respectively, in points 1 and 2, as shown in the accompanying figure. We suppose that the functions defining $C_1$ and $C_2$ have continuous first and second derivatives and that these arcs have no singular points. The arc $E$ is continuous and consists of a finite number of pieces each of which has a continuously turning tangent.

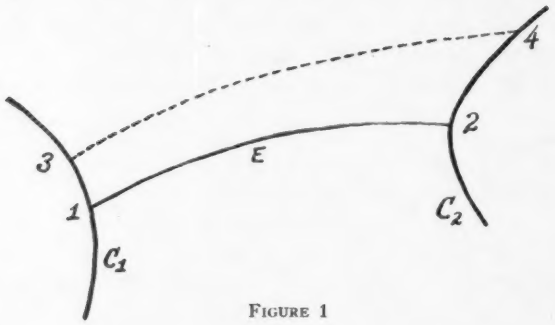


FIGURE 1

Consider now an integral

$$I = \int_{x_1}^{x_2} f(x, y, y')dx$$

whose integrand function $f(x, y, y')$ has continuous partial derivatives up to and including those of the fourth order for all sets of values $(x, y, y')$ in a neighborhood of those on $E$. The arcs which have continuity properties similar to those of $E$ and whose elements all lie in this neighborhood may be called admissible arcs. The problem before us is to discuss the necessary conditions which $E$ must satisfy if it minimizes $I$ in the class of admissible arcs joining $C_1$ and $C_2$.

It is well known that an equation $f_{y'} = \int_a^x f_y\, dx + c$ must be satisfied by $E$,* from which it follows that the function $f_{y'}$ is

necessarily continuous, when considered as a function of $x$ along $E$, even at the corners of $E$, and that the Euler equation

$$\frac{d}{dx} f_{y'} = f_y \tag{1}$$

is satisfied on every sub-arc of $E$ on which the tangent turns continuously. Furthermore at the points 1 and 2 the transversality condition

$$f(x,y,y')X' + (Y' - y'X')f_{y'}(x,y,y') = 0 \tag{2}$$

must be satisfied by the element $(x,y,y')$ of $E$ and the direction $X' : Y'$ of the arc $C_1$ or $C_2$.† We assume, as is customary for problems with variable end-points in the calculus of variations, that the value of $f$ at the points 1 and 2 on $E$ is different from zero. Since $C_1$ and $C_2$ have no singular points it follows from this assumption and equation (2) that the difference $Y' - y'X'$ is not zero at the points 1 and 2, or in other words that $E$ is not tangent to either $C_1$ or $C_2$.

In the discussion of further necessary conditions for a minimum we may agree to limit ourselves to arcs $E$ for which the function $y(x)$ has continuous first and second derivatives on an interval containing $x_1x_2$, and we may consider as usual a one-parameter family of arcs of the form

$$y = y(x) + a\eta(x) = y(x,a) \tag{3}$$

where $\eta$ has continuity properties like those just prescribed for $y(x)$ but is otherwise arbitrary. The equations

$$Y_1(u_3) - y(X_1(u_3),a) = 0\,, \quad Y_2(u_4) - y(X_2(u_4),a) = 0 \tag{4}$$

determine the parameter values $u_3$ and $u_4$ where the arc (3) intersects $C_1$ and $C_2$, respectively, as shown in Figure 1. The former, for example, has a solution $(u_3,a) = (u_1,0)$, corresponding to the point 1 of the figure, at which the derivative of its first member with respect to $u_3$ is

$$Y_1'(u_1) - y_x(X_1'(u_1),0)X_1'(u_1) = Y_1'(u_1) - y'(x_1)X_1'(u_1) \neq 0\,.$$

* Bliss, *Calculus of Variations*, p. 130.

† Bliss, loc. cit., p. 167.

It follows from well known implicit function theorems that the first equation (4) has a unique solution $u_3(a)$ with continuous first and second derivatives near $a=0$ and with the initial value $u_3(0)=u_1$. The second has a similar solution $u_4(a)$ with initial value $u_4(0)=u_2$. The derivatives at $a=0$ of these two functions are both defined by the equation

$$\frac{du}{da}=\frac{y_a(X,0)}{Y'-y_x(X,0)X'}=\frac{\eta(x)}{Y'-y'(X)X'} \tag{5}$$

when suitable subscripts are attached to $X$ and $Y$, and when the arguments $u_1$ and $u_2$, respectively, are substituted. The coordinates of the intersection points 3 and 4 are also functions of $a$ defined by the equations

$$x_3(a)=X_1(u_3(a)), \qquad y_3(a)=y(x_3(a),a)=Y_1(u_3(a)),$$

$$x_4(a)=X_2(u_4(a)), \qquad y_4(a)=y(x_4(a),a)=Y_2(u_4(a)),$$

and their derivatives satisfy the relations

$$\left\{\begin{aligned} &\frac{dx}{da}=X'\frac{du}{da}, \qquad \frac{d^2x}{da^2}=X'\frac{d^2u}{da^2}+X''\left(\frac{du}{da}\right)^2,\\ &\frac{dy}{da}=y_x\frac{dx}{da}+y_a=Y'\frac{du}{da},\\ &\frac{d^2y}{da^2}=y_x\frac{d^2x}{da^2}+y_{xx}\left(\frac{dx}{da}\right)^2+2y_{xa}\frac{dx}{da}=Y'\frac{d^2u}{da^2}+Y''\left(\frac{du}{da}\right)^2. \end{aligned}\right. \tag{6}$$

If the arc $E$ minimizes $I$ as described above the function

$$I(a)=\int_{x_3}^{x_4} f(x,y(x,a),y_x(x,a))\,dx$$

taken along the arc (3) from its intersection 3 with $C_1$ to its intersection 4 with $C_2$ must have a minimum at the value $a=0$, and its derivatives must satisfy the conditions $I'(0)=0$, $I''(0)\geqq 0$. The values of these derivatives are readily found to be

$$I'(a)=f\frac{dx}{da}\Big|_3^4+\int_{x_3}^{x_4}(f_y y_a+f_{y'}y_{xa})\,dx,$$

$$I''(a)=f\frac{d^2x}{da^2}+(f_z+f_y y_z+f_{y'}y_{xz})\left(\frac{dx}{da}\right)^2$$
$$+2(f_y y_a+f_{y'}y_{xa})\frac{dx}{da}\Big|_3^4+\int_{x_3}^{x_4}2\omega(x,y_a,y_{ax})dx$$

where $2\omega(x,\eta,\eta')=f_{yy}\eta^2+2f_{yy'}\eta\eta'+f_{y'y'}\eta'^2$. When $a=0$, the first of these vanishes, as may be shown by integrating the second term of the integrand by parts and applying Euler's equation (1) and the transversality conditions (2). With the help of the last equation (6) the second derivative at $a=0$ takes the form

$$I''(0)=(f-y_z f_{y'})\frac{d^2x}{da^2}+f_{y'}\frac{d^2y}{da^2}+(f_z+y_z f_y)\left(\frac{dx}{da}\right)^2$$
$$+2f_y y_a\frac{dx}{da}\Big|_1^2+\int_{x_1}^{x_2}2\omega(x,y_a,y_{ax})dx\,.$$

When the partial derivatives $y_a=\eta$ and the total derivatives of $x$ and $y$ with respect to $a$ are substituted from equations (6) and (5) it is found after some computation that this derivative has the value

$$I''(0)=L\eta^2\Big|_1^2+\int_{x_1}^{x_2}2\omega(x,\eta,\eta')dx \tag{7}$$

where $L$ is defined by the equations

$$L=\frac{M}{r}+N\,,\quad M=[X'f_{y'}-Y'(f-y'f_{y'})]\frac{\sqrt{X'^2+Y'^2}}{(Y'-y'X')^2}\,,$$
$$N=[(f_z+f_y y')X'+2f_y(Y'-y'X')]\frac{X'}{(Y'-y'X')^2}\,,$$

in which $r$ is the radius of curvature of $C_1$ or $C_2$. The only term which might cause difficulty in this computation is the term $M/r$ in $L$. To find it we note with the help of equations (6) that

$$(f-y'f_{y'})\frac{d^2x}{da^2}+f_{y'}\frac{d^2y}{da^2}=[(f-y'f_{y'})X'+f_{y'}Y']\frac{d^2u}{da^2} \tag{8}$$
$$+[(f-y'f_{y'})X''+f_{y'}Y'']\left(\frac{du}{da}\right)^2.$$

The coefficient of $d^2u/da^2$ in this expression vanishes at 1 and 2 on account of the transversality condition (2), and it follows from the same condition that at those points

$$f-y'f_{y'}=-kY', \qquad f_{y'}=kX', \qquad k=\frac{X'f_{y'}-Y'(f-y'f_{y'})}{X'^2+Y'^2}.$$

With the help of the value (5) for $du/da$ we now find that the value of the expression (8) is $(M/r)\eta^2$. The other terms in the expression for $I''(0)$ give no trouble, and we may summarize the results of this section as follows.

THEOREM 1. *If the arc $E$ is to minimize the integral $I$ in a class of neighboring arcs joining the curves $C_1$ and $C_2$, as described above, then the expression (7) for the second variation $I''(0)$ must be $\geqq 0$ for every function $\eta(x)$ having continuous first and second derivatives on the interval $x_1x_2$.*

One can infer readily that $I''(0)$ must also be $\geqq 0$ for all continuous functions $\eta(x)$ on $x_1x_2$ which have continuous first derivatives except possibly at a finite number of corners. This can be done by modifying suitably the analysis of the preceding paragraphs, or by showing that every function $\eta(x)$ of this character can be approximated by a polynomial giving $I''(0)$ a value near to that for $\eta(x)$.*

3. *The Problem of Minimizing the Second Variation.* Let us consider in this section the problem of determining when an expression of the form

$$I_2(\eta)=\int_{x_1}^{x_2} 2\Omega(x,\eta,\eta')dx$$

is $\geqq 0$ in the class $H$ of functions $\eta(x)$ continuous on $x_1x_2$ and having continuous derivatives except possibly at a finite number of corners. The integrand $2\Omega$ is understood to be a quadratic form

$$2\Omega(x,\eta,\eta')=P(x)\eta^2+2Q(x)\eta\eta'+R(x)\eta'^2$$

with $R>0$ on $x_1x_2$ and with $P, Q, R$ having continuous first

* Hadamard, *Calcul des Variations*, pp. 51–54.

derivatives on this interval. The second variation (7) is easily seen to be an integral of the type $I_2(\eta)$ if we set

$$2\Omega = \frac{d}{dx} L\eta^2 + f_{vv}\eta^2 + 2f_{vv'}\eta\eta' + f_{v'v'}\eta'^2 , \tag{9}$$

where $L$ is a function of $x$ with continuous second derivatives on $x_1x_2$ and with $L_1$ and $L_2$ as initial and end-values on this interval.

The problem of minimizing $I_2(\eta)$ in the class $H$ of functions $\eta(x)$ is that of finding a curve in the $x\eta$-plane joining the two ordinates $x=x_1$ and $x=x_2$ and minimizing $I_2(\eta)$. It is therefore of the same type as the original problem of the last section in the $xy$-plane, and it is clear that a minimizing curve for the $x\eta$-problem must satisfy an equation

$$\Omega_{\eta'} = \int_{x_1}^{x} \Omega_\eta dx + c$$

and the transversality conditions

$$\Omega_{\eta'}|^1 = 0 , \qquad \Omega_{\eta'}|^2 = 0 . \tag{10}$$

If we substitute the derivatives of the quadratic form $\Omega$ in these three equations we find from the first that a minimizing function $\eta(x)$ must have continuous first and second derivatives and satisfy the Jacobi equation

$$\Omega_\eta - \frac{d}{dx}\Omega_{\eta'} = (P-Q')\eta - R'\eta' - R\eta'' = 0 \tag{11}$$

where $\Omega$ is defined by the equation preceding (9). This is identical with the Jacobi equation deduced from the quadratic form $2\omega$ in the second variation (7) when $2\Omega$ has the value (9), as one readily verifies. The equations (10) show that a minimizing function must further satisfy the boundary conditions

$$Q_1\eta(x_1) + R_1\eta'(x_1) = 0 , \qquad Q_2\eta(x_2) + R_2\eta'(x_2) = 0 ,$$

where the numerical subscripts designate the values of $Q(x)$, $R(x)$ at $x_1$ or $x_2$. One can readily interpret these conditions in terms of the coefficients of the quadratic form $2\omega$

and the constants $L_1$ and $L_2$ of the second variation when $2\Omega$ has the value (9). We have then the following result.

THEOREM 2. *A function $\eta(x)$ which minimizes $I_2(\eta)$ in the class H must have continuous first and second derivatives and be a solution of the boundary value problem*

$$J(\eta)=0\,, \qquad Q_1\eta(x_1)+R_1\eta'(x_1)=Q_2\eta(x_2)+R_2\eta'(x_2)=0\,,$$

*where $J(\eta)$ is a symbol for the first member of the Jacobi equation* (11).

We shall need presently the following lemma.

LEMMA 1. *If $u(x)$ is a solution of the equation $J(u)=0$ and is different from zero on $x_1x_2$ then for every $\eta(x)$ in the class H*

$$(12)\qquad \int_{x_1}^{x_2} 2\Omega dx=\frac{\eta^2}{u}\,\Omega_{u'}\Big|_1^2+\int_{x_1}^{x_2}R(x)\left(\eta'-\frac{\eta}{u}u'\right)^2dx$$

*where $\Omega_{u'}=\Omega_{\eta'}(x,u,u')$.*

To prove this consider the function $a(x)$ and its derivative determined by the equations

$$\eta(x)=a(x)\,u(x)\,, \qquad \eta'=a\,u'+a'u\,.$$

With the help of Taylor's formula and the equation $J(u)=0$ we see that

$$\begin{aligned}2\Omega(x,\eta,\eta')&=2\Omega(x,au,au')+2a'u\,\Omega_{\eta'}(x,au,au')+Ra'^2u^2\\&=a^2(u\Omega_u+u'\Omega_{u'})+2aa'u\Omega_{u'}+Ra'^2u^2\\&=\frac{d}{dx}a^2u\Omega_{u'}+R(\eta'-au')^2\\&=\frac{d}{dx}\frac{\eta^2}{u}\,\Omega_{u'}+R\left(\eta'-\frac{\eta}{u}u'\right)^2.\end{aligned}$$

An integration gives at once the formula of the lemma.

THEOREM 3. *A necessary and sufficient condition that the relation $I_2(\eta)\geqq 0$ shall hold for every function $\eta(x)$ of the class H is that the two solutions $u_1(x)$, $u_2(x)$ of the equation $J(u)=0$ defined by the conditions*

$$u_1(x_1)=R_1\,,\qquad u_1'(x_1)=-Q_1\,,$$
$$u_2(x_2)=R_2\,,\qquad u_2'(x_2)=-Q_2\,, \tag{13}$$

*be positive and have*

$$u_2u_1'-u_1u_2'>0 \tag{14}$$

*on the interval* $x_1x_2$.

To prove the necessity we note first that for every solution $u(x)$ of the equation $J(u)=0$ we have

$$2\Omega(x,u,u')=u\Omega_u+u'\Omega_{u'}=\frac{d}{dx}u\Omega_{u'}\,. \tag{15}$$

The function $u_1(x)$ is positive at $x=x_1$ and can not vanish at any value $\xi$ between $x_1$ and $x_2$. Otherwise we could define $\eta(x)$ by the equations

$$\eta(x)=u_1(x)\quad \text{for}\quad x_1\leqq x\leqq\xi\,,$$
$$=0\quad \text{for}\quad \xi\leqq x\leqq x_2\,,$$

and for this function we should find with the help of (15) the value

$$I_2(\eta)=u_1\Omega_{u_1'}\Big|_{x_1}^{\xi}=0$$

since $\Omega_{u_1'}$ vanishes at $x_1$ and $u_1$ at $\xi$. But this function $\eta(x)$ would have a discontinuous derivative at $x=\xi$ since a solution $u_1(x)$ of $J(u)=0$ can vanish with its derivative only when identically zero. Hence $\eta(x)$ could not minimize $I_2(\eta)$ and there would be $\eta$'s giving $I_2(\eta)$ negative values. In the same manner we may prove that $u_2(x)$ does not vanish between $x_1$ and $x_2$ when the values of $I_2(\eta)$ are all $\geqq 0$ in $H$.

The equation

$$0=u_1J(u_2)-u_2J(u_1)=\frac{d}{dx}R(u_2u_1'-u_1u_2')$$

shows that

$$R(u_2u_1'-u_1u_2')=c \tag{16}$$

and hence that the determinant (14) has always the same

sign or else vanishes identically on $x_1x_2$. We may define $\eta(x)$ by the equations

$$\eta(x)=u_2(\xi)u_1(x) \quad \text{for} \quad x_1\leqq x\leqq \xi\,,$$
$$=u_1(\xi)u_2(x) \quad \text{for} \quad \xi\leqq x\leqq x_2,$$

and the value of $I_2(\eta)$, with the help of (15), is then

$$I_2(\eta)=u_1u_2\left(u_2\Omega_{u_1'}-u_1\Omega_{u_2'}\right)|^{\xi}=cu_1(\xi)u_2(\xi)$$

which would be negative, since $u_1$ and $u_2$ are both positive between $x_1$ and $x_2$, if the constant $c$ in equation (16) were negative. Hence we see that the determinant (14) can not be negative when $I_2(\eta)\geqq 0$ in $H$.

The function $u_1(x)$ can not vanish even at $x=x_2$ since if $u_1(x_2)$ were zero we should have $u_1'(x_2)<0$ and the determinant (14) would be negative since $u_2(x_2)$ is positive. Similarly $u_2(x)$ can not vanish at $x_1$.

To prove the sufficiency of the condition of the theorem let $\eta(x)$ be an arbitrary function of the class $H$ and let us apply the formula (12) to $\eta$ and $u_1$ on an interval $x_1\xi$, and to $\eta$ and $u_2$ on the remaining interval $\xi x_2$. Then

$$I_2(\eta)=\frac{\eta^2}{u_1}\Omega_{u_1'}-\frac{\eta^2}{u_2}\Omega_{u_2'}\Big|^{\xi}+\int_{x_1}^{x_2}R\left(\eta'-\frac{\eta}{u}u'\right)^2dx$$

$$=\frac{c\eta^2(\xi)}{u_1(\xi)u_2(\xi)}+\int_{x_1}^{x_2}R\left(\eta'-\frac{\eta}{u}u'\right)^2dx \tag{17}$$

where $u$ in the last integral is $u_1$ on $x_1\xi$, and $u_2$ on $\xi x_2$. The last expression is evidently not negative under the hypothesis of the theorem.

COROLLARY 1. *A necessary and sufficient condition that the relation $I_2(\eta)>0$ shall hold for every function of $H$ except $\eta(x)\equiv 0$ is that $u_1(x)$ and $u_2(x)$ be positive and have*

$$u_2u_1'-u_1u_2'>0 \tag{18}$$

*on the interval $x_1x_2$.*

To prove the necessity we observe that the last theorem requires $u_1$ and $u_2$ to be positive and $u_2u_1'-u_1u_2'$ to be $\geqq 0$

on $x_1x_2$. But if this determinant were zero we should have $u_1 = ku_2$, and then $\eta = u_1$ would make $I_2(\eta) = 0$, as we readily see by applying equations (15) and (13). Hence $I_2(\eta)$ would vanish for an $\eta(x) \not\equiv 0$ in $H$ unless the condition (18) is satisfied.

To prove the sufficiency we note that the first term in the expression (17) is certainly positive for some value of $\xi$, under the hypotheses of the theorem, unless $\eta(x) \equiv 0$ on $x_1x_2$.

COROLLARY 2. *If the condition of the last corollary holds then there is a constant $\delta > 0$ such that $I_2(\eta) > \delta$ for every function $\eta(x)$ of the class $H$ satisfying the equation*

$$\int_{x_1}^{x_2} \eta^2(x)dx = 1 . \tag{19}$$

This follows from the formula (17) since for such an $\eta(x)$ there must be at least one point $\xi$ on the interval $x_1x_2$ where $\eta^2(\xi) > 1/(x_2 - x_1)$. The corollary is analogous to the theorem of Osgood cited on a preceding page.

Let us define $H_1$ as the class of all functions of $H$ which satisfy the equation (19). Then it is of interest to note the following theorem.

THEOREM 4. *A function $\eta(x)$ which minimizes $I_2(\eta)$ in the class $H_1$ is necessarily a solution of the boundary value problem*

$$J(\eta) - \lambda\eta = 0 ,$$
$$Q_1\eta(x_1) + R_1\eta'(x_1) = Q_2\eta(x_2) + R_2\eta'(x_2) = 0 . \tag{20}$$

For suppose that $\eta(x)$ is such a minimizing function and consider the two-parameter family of functions $\eta + a\zeta + a_1\zeta_1$ where $\zeta$ and $\zeta_1$ are two arbitrarily selected functions of $H$. Denote by $I(a, a_1)$, $K(a, a_1)$ the values of $I_2$ and the integral in (19) on an arc of this family. The determinant

$$I_a(0,0)K_{a_1}(0,0) - I_{a_1}(0,0)K_a(0,0)$$

must be zero for every choice of $\zeta$ and $\zeta_1$. Otherwise, according to well known implicit function theorems, the equations

$$I(a, a_1) = I(0,0) + u , \quad K(a, a_1) = 1$$

would have solutions for both positive and negative values of $u$ near $u=0$, and $I(0,0)$ could not be a minimum value of $I(a,a_1)$. Since the determinant vanishes for every choice of $\zeta$ and $\zeta_1$ let us fix $\zeta_1$ and choose $\lambda$ so that the second of the equations

$$I_a(0,0)-\lambda K_a(0,0)=0, \quad I_{a_1}(0,0)-\lambda K_{a_1}(0,0)=0$$

is satisfied. Then the first must hold for every $\zeta$ in the class $H$. It has the form

$$I_a(0,0)-\lambda K_a(0,0)=\int_{x_1}^{x_2} 2[\Omega_\eta\zeta+\Omega_{\eta'}\zeta'-\lambda\eta\zeta]dx=0\ .$$

Since this must be true for every $\zeta$ in $H$ it follows, by the customary integration of the second term of the integrand by parts, that $\eta$ satisfies the conditions (20), which was to be proved.

It is evident that if the expression $I_2(\eta)$ is $\geqq 0$ in the class $H$ it will also have this property in the class $H_1$, and conversely. We shall in the next section be concerned with the problem of determining the minimum of $I_2(\eta)$ in the class $H_1$.

4. *The Boundary-Value Problem.* From the theorems of the preceding section it may be inferred that the solution of the boundary value problem

$$J(\eta)-\lambda\eta=0, \tag{21}$$
$$Q_1\eta(x_1)+R_1\eta'(x_1)=Q_2\eta(x_2)+R_2\eta'(x_2)=0$$

is closely related to the problem of determining the minimum of the integral

$$I_2(\eta)=\int_{x_1}^{x_2} 2\Omega(x,\eta,\eta')dx$$

in the class $H_1$ of functions $\eta(x)$ satisfying the equation

$$\int_{x_1}^{x_2}\eta^2(x)dx=1\ ,$$

where $2\Omega=P(x)\eta^2+2Q(x)\eta\eta'+R(x)\eta'^2$. It is the purpose of this section to investigate this relationship further.

THEOREM 5. *The values of the integral $I_2(\eta)$ have a greatest lower bound $\lambda_1$ in the class $H_1$.*

For we may readily verify that

$$I_2(\eta)-\lambda\int_{x_1}^{x_2}\eta^2dx=\int_{x_1}^{x_2}\left[R\left(\frac{Q}{R}\eta+\eta'\right)^2+\left(P-\frac{Q^2}{R}-\lambda\right)\eta^2\right]dx.$$

By taking $\lambda$ negative and sufficiently large the coefficient of $\eta^2$ in the last integral will be positive on $x_1x_2$ and the integral itself will be positive for every function $\eta(x)$ in $H$ not identically zero. From the first member of the equation it follows then that $I_2(\eta)$ will be greater than $\lambda$ for every function $\eta(x)$ in $H_1$.

THEOREM 6. *The boundary value problem* (20) *has a solution for* $\lambda=\lambda_1$, *and this is the smallest value of* $\lambda$ *for which such a solution exists.*

It is provable readily that no solution of the boundary value problem exists for a value $\lambda<\lambda_1$. Such a solution would give $I_2(\eta)$ the value

$$I_2(\eta)=\int_{x_1}^{x_2}(\eta\Omega_\eta+\eta'\Omega_{\eta'})dx=\eta\Omega_{\eta'}\Big|_1^2+\int_{x_1}^{x_2}\eta J(\eta)dx=\lambda\int_{x_1}^{x_2}\eta^2dx,$$

since $\Omega_{\eta'}$ vanishes at $x_1$ and $x_2$ and $J(\eta)=\lambda\eta$. By multiplying $\eta(x)$ by a suitable constant a function of $H_1$ would be found giving $I_2(\eta)$ the value $\lambda$ which is less than the minimum $\lambda_1$.

The expression

$$I_2(\eta)-\lambda\int_{x_1}^{x_2}\eta^2dx \tag{22}$$

is always $\geqq 0$ in $H_1$ when $\lambda=\lambda_1$, and hence also in $H$, as one easily verifies. By Theorem 3 of the preceding section it follows that the solutions $u_1(x,\lambda)$, $u_2(x,\lambda)$ of the equation $J(u)-\lambda u=0$, determined by the conditions (13), must for $\lambda=\lambda_1$ be positive and satisfy the relation

$$u_2u_1'-u_1u_2'\geqq 0 \tag{23}$$

on the interval $x_1x_2$. If the inequality were true then Corollary 2 of Theorem 3 shows that in the class $H_1$ the expression (22) for $\lambda=\lambda_1$ would always exceed a constant $\delta>0$, or in other words that $I_2(\eta)>\lambda_1+\delta$ in $H_1$. But this is impossible when $\lambda_1$ is the minimum described above. Hence the equality sign in the relation (23) must hold. But in that case $u_1=ku_2$ and $u_1$ is therefore a solution of the boundary value problem (20).

COROLLARY 1. *A necessary and sufficient condition that the relation* $I_2(\eta)\geqq 0$ *holds for every function* $\eta(x)$ *of the class H is that* $\lambda_1\geqq 0$ *where* $\lambda_1$ *is the smallest value for which the boundary value problem* (20) *has a solution. A necessary and sufficient condition that* $I_2(\eta)>0$ *for every function* $\eta(x)$ *in H except* $\eta(x)\equiv 0$ *is* $\lambda_1>0$.

This conclusion is an immediate consequence of the fact that $\lambda_1$ is the minimum of the values of $I_2(\eta)$ in the class $H_1$.

COROLLARY 2. *Let E be an arc, as described in Section* 1, *which gives the integral I there considered the value I(E), and suppose that along E the derivative* $f_{y'y'}$ *is positive. Then a necessary condition for I(E) to be a minimum is that* $\lambda_1\geqq 0$ *where* $\lambda_1$ *is the smallest value of* $\lambda$ *for which the boundary value problem* (20) *associated with the second variation has a solution.*

It should be remarked that the necessary condition $\lambda_1\geqq 0$ is equivalent to the well known analog of Jacobi's condition for this calculus of variations problem.* The focal points of the curves $C_1$ and $C_2$ on $E$ are determined by the roots of $u_1(x,0)$ and $u_2(x,0)$, as one readily verifies. The condition $\lambda_1\geqq 0$ implies that the integral $I_2(\eta)$ is $\geqq 0$ for all functions $\eta$ in $H$ when $\lambda=0$, and hence, by Theorem 3, that the roots of $u_1(x,0)$ and $u_2(x,0)$ do not lie on $x_1x_2$. This means that the focal points 1′ and 2′ of $C_1$ and $C_2$ do not lie on the arc 12 of $E$. Furthermore the condition $u_2u_1'$

* See Bliss, MATHEMATISCHE ANNALEN, vol. 58 (1904), p. 70.

$-u_1u_2'>0$ implies that the derivative $d(u_2/u_1)/dx$ is less than or equal to zero, so that the equation $u_1=0$ can have no root $x_1'>x_2$ preceding the first root $x_2'$ of $u_2=0$. The points 1 and 1′ are therefore surely not separated by 2 and 2′ on the arc $E$. This is the complete Jacobi condition as described in the reference of the footnote on the preceding page.

UNIVERSITY OF CHICAGO

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# A CONNECTED AND REGULAR POINT SET WHICH CONTAINS NO ARC*

BY R. L. MOORE

A point set is said to be *connected im kleinen*,† or *regular*, at the point $P$ if, for every positive number $e$, there exists a positive number $d_e$ such that if $X$ is any point of $M$ at a distance from $P$ less than $d_e$ then $X$ and $P$ lie together in some connected‡ subset of $M$ of diameter less than $e$. A point set which is regular (connected im kleinen) at every one of its points is said to be regular (connected im kleinen). The set $M$ is *uniformly* connected im kleinen if for each positive number $e$ there exists a positive number $d_e$ such that every two points of $M$ at a distance apart less than $d_e$ lie in a connected subset of $M$ of diameter less than $e$. If a point set $M$ is con-

* Presented to the Society, September 6, 1923.

† Cf. Hans Hahn, *Ueber die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist*, JAHRESBERICHT DER VEREINIGUNG, vol. 23 (1914), pp. 318–322. Also S. Mazurkiewicz, *Sur les lignes de Jordan*, FUNDAMENTA MATHEMATICAE, vol. 1 (1920), pp. 166–209. This conception, as applied to a simple closed curve, was used by Pia Nalli in the paper *Sopra una definizio ni di dominio piano limitato da una curva continua, senza punti multipli*, RENDICONTI DI PALERMO, vol. 32 (1911), pp. 391–401.

‡ According to Hahn's formulation, $X$ and $P$ lie in a *closed* and connected subset of $M$ of diameter less than $e$. It has been customary with me to omit the stipulation that this subset should be closed. However, the set $M$ described below is connected im kleinen according to either definition.

nected, connected im kleinen and *closed*, then every two of its points can* be joined by a simple continuous arc which lies wholly in $M$. In the present note I will show that there exists a connected and connected im kleinen point set which is not closed and which contains no arc whatsoever.

Let $S$ denote a unit square. For each positive integer $n$, divide $S$ into $n^2$ equal squares and let $S_n$ denote this set of $n^2$ squares. For each square $q$ of the set $S_n$ construct just one closed and connected point set † $K_q$ which contains no arc and which lies wholly within $q$ except that it contains just four points on $q$, these four points being the mid-points of the four sides of $q$. For each $n$ let $M_n$ denote the set of points obtained by adding together the point sets $K_q$ for all squares $q$ of the set $S_n$. Let $M$ denote the point set

$$M_1+M_2+M_3+\cdots.$$

This point set is connected and uniformly connected im kleinen. But it contains no arc since it is the sum of a countable number of closed point sets no one of which contains an arc.

THE UNIVERSITY OF TEXAS

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* R. L. Moore, *A theorem concerning continuous curves*, this BULLETIN, vol. 23 (1917), pp. 233–236; S. Mazurkiewicz, loc. cit.; H. Tietze, *Ueber stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven*, MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), pp. 284–291.

† That such a set exists may be easily seen with the help of an example described by B. Knaster in his paper *Un continu dont tout sous-continu est indécomposable*, FUNDAMENTA MATHEMATICAE, vol. 3 (1922), pp. 247–286.

# THE ALTERNATION OF NODES OF LINEARLY INDEPENDENT SOLUTIONS OF SECOND ORDER DIFFERENCE EQUATIONS*

BY OTTO DUNKEL

We shall consider solutions of the difference equation

$$(1) \qquad u(n+2)=A(n)\,u(n+1)-B(n)\,u(n), \qquad B(n)>0,$$

where $A(n)$ and $B(n)$ are finite and single-valued functions of the integer $n$. If the points obtained by plotting a solution $u(n)$ are joined by segments of a straight line, this broken line gives a representation of a single-valued and continuous function $f(x)$ such that $f(n)=u(n)$. The zeros of $f(x)$ are called the nodes of $u(n)$.

Proofs have already been given of the following theorem.

THEOREM. *The nodes of two linearly independent solutions of* (1) *separate one another.*†

The proof which is to be given here seems simpler and more obvious than either of these two proofs. Two known and easily verified facts will be used. If $u_1(n)$ and $u_2(n)$ are any two solutions of (1) and if we set

$$\Delta(n)=\begin{vmatrix} u_1(n) & u_2(n) \\ u_1(n+1) & u_2(n+1) \end{vmatrix},$$

then

$$\Delta(n+1)=B(n)\Delta(n)\ .$$

As a first result of the condition imposed upon $B(n)$ in (1) we have the fact that if $\Delta(n)$ is not zero for one value of $n$ then it is never zero and its sign remains unchanged.

A necessary and sufficient condition that the two solutions $u_1(n)$ and $u_2(n)$ are linearly independent is that $\Delta(n)$ is not zero for one value of $n$.

* Presented to the Society, December 29, 1925.

† Porter, ANNALS OF MATHEMATICS, (2), vol. 3, (1901–02), p. 65. Moulton, E. J., ibid. ,(2), vol. 13 (1911–12), p. 137.

We have

$$f_i(x)=(x-n)[u_i(n+1)-u_i(n)]+u_i(n), \quad n\leqq x<n+1, \quad i=1,2 .$$

An easy calculation shows that

$$W(x)=\begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}=\Delta(n) , \qquad n<x<n+1 . \tag{2}$$

If we set

$$f_i'(n)=u_i(n+1)-u_i(n) , \tag{3}$$

then $W(x)$ is defined for all values of $x$, and it is never zero and it has always the same sign, if, as we shall now suppose, $u_1(n)$ and $u_2(n)$ are linearly independent.

We shall now have to notice a second result of the condition placed upon $B(n)$. If $u(n+1)=0$, $u(n+2)=-B(n)u(n)$, and hence $f'(x)$, which may be discontinuous at $n+1$, has the same sign a little before, at, and a little after $n+1$. The case in which $u(n)$ is zero for two successive values of $n$ cannot occur here for two linearly independent solutions.

Suppose that $f_1(x)$ vanishes at $x_1$ and at $x_2$ but at no point between. Then

$$f_1'(x_1)f_1'(x_2)<0 , \tag{4}$$

even if one or both of $x_1$ and $x_2$ are integers as a result of the remark above. Then from the facts stated above

$$W(x_1)W(x_2)=f_1'(x_1)f_2(x_1)f_1'(x_2)f_2(x_2)>0 . \tag{5}$$

From (4) and (5) it follows that

$$f_2(x_1)f_2(x_2)<0 ,$$

and this requires that $f_2(x)$, which is continuous, shall vanish at least once between $x_1$ and $x_2$. It cannot vanish more than once, for if it did then the same form of proof would show that $f_1(x)$ vanishes between $x_1$ and $x_2$ contrary to the hypothesis. This concludes the proof.

WASHINGTON UNIVERSITY

# A NOTION OF UNIFORM INTEGRABILITY*

BY R. E. LANGER AND J. D. TAMARKIN

The necessary and sufficient condition that a function $f(x)$ of the real variable $x$ be integrable in the sense of Riemann on the interval $(a,b)$ is that there correspond to an arbitrary small positive number $\epsilon$ a positive $\delta$ such that for any subdivision of $(a,b)$ by points

$$x_0=a\leqq x_1\leqq x_2\leqq \cdots \leqq x_{n-1}\leqq x_n=b\ ,$$

subject to the condition $x_i-x_{i-1}<\delta$, the inequality

$$\sum_{i=1}^{n}(U_i-L_i)(x_i-x_{i-1})\leqq\epsilon$$

is valid. In this, $U_i$ and $L_i$ represent, respectively, the upper and lower bounds of $f(x)$ on the subinterval $(x_{i-1},x_i)$.

In the direct extension of this definition to a function which involves besides the variable of integration also other parameters, it may or may not be possible in any particular case to satisfy the conditions above by a constant $\delta$ independent of the parameters. In this connection the following concept may be of interest.

A function $f(x,\lambda)$ shall be defined to be integrable with respect to $x$ on $(a,b)$ uniformly in $\lambda$, provided that there corresponds to an arbitrary positive $\epsilon$ a positive constant $\delta$ independent of $\lambda$, such that

$$\sum_{i=1}^{n}\{U_i(\lambda)-L_i(\lambda)\}(x_i-x_{i-1})\leqq\epsilon.\dagger \tag{1}$$

If $f(x,\lambda)$ is complex, it shall be said to be uniformly integrable if both its real and its imaginary parts are uniformly integrable.

* Presented to the Society, May 1, 1926.

† The extension of this definition to the case when $f$ involves a greater number of parameters, real or complex, is, of course, immediate.

As an application of this notion of uniform integrability we establish the following theorem.

THEOREM. *If the function $f(x,\lambda)$ of the real variable $x$ and the complex parameter $\lambda$ is, in the region*

$$0 \leqq a \leqq x \leqq b, \qquad R(\lambda) \leqq M \geqq 0,^*$$

*defined, uniformly bounded, and integrable with respect to $x$, uniformly in $\lambda$, then the integral*

$$I(f) = \int_a^b e^{\lambda x} f(x,\lambda)\,dx$$

*approaches zero uniformly in $\lambda$ as $|\lambda|$ becomes infinite.*

It is sufficient for our purpose to give the proof for the real part only of the given function, $f^{(1)}(x,\lambda)$. Because of the uniform integrability of $f^{(1)}(x,\lambda)$, we may, when $\epsilon$ is assigned, determine $\delta$ so that (1) is satisfied. Then choosing on each subinterval $(x_{i-1},x_i)$ a value $P_i(\lambda)$ subject to the conditions

$$L_i(\lambda) \leqq P_i(\lambda) \leqq U_i(\lambda),$$

and defining the auxiliary stepfunction $\psi(x,\lambda)$ as the function taking the value $P_i(\lambda)$ on $(x_{i-1},x_i)$ we have

$$I(f^{(1)}) = I(\psi) + I(f^{(1)} - \psi).$$

Now on the one hand

$$|I(\psi)| = \left| \sum_{i=1}^{n} P_i(\lambda) \cdot \frac{e^{\lambda x_i} - e^{\lambda x_{i-1}}}{\lambda} \right|,$$

whence, since $|P_i(\lambda)| \leqq k$ because $f^{(1)}$ is uniformly bounded,

$$|I(\psi)| \leqq \frac{2kne^{Mb}}{|\lambda|}.$$

On the other hand since $f^{(1)}$ is uniformly integrable

$$|I(f^{(1)} - \psi)| \leqq e^{Mb} \int_a^b |f^{(1)}(x,\lambda) - \psi(x,\lambda)|\,dx$$

$$\leqq e^{Mb} \sum_{i=1}^{n} \{U_i(\lambda) - L_i(\lambda)\}(x_i - x_{i-1}) \leqq e^{Mb}\epsilon.$$

* If these restrictions on $x$ and $\lambda$ are omitted, the reasoning employed leads to the result that $I(f) = e^{\lambda a}\epsilon(\lambda) + e^{\lambda b}\epsilon(\lambda)$, where each $\epsilon(\lambda)$ denotes some function which approaches zero uniformly as $\lambda \to \infty$. The symbol $R(\lambda)$ designates the real part of $\lambda$.

Hence

$$|I(f^{(i)})| \leqq e^{Mb}\left[\frac{2nk}{|\lambda|}+\epsilon\right],$$

and since $n$ is fixed when $\epsilon$ is given, the theorem is proved.

The situation changes materially when integrability is considered in the sense of Lebesgue. By definition, then, the integral of $f$ is equal to the limit as $\delta \to 0$ of the series

$$\sum_{k=-\infty}^{\infty} y_k m E_x(y_{k-1} < f(x,\lambda) \leqq y_k), \tag{2}$$

where $\cdots y_{-2} < y_{-1} < y_0 < y_1 < y_2 \cdots$ denotes an arbitrary subdivision of the range of functional values subject to the condition $y_k - y_{k-1} < \delta$. It would seem natural to define the integrability of $f(x,\lambda)$ as uniform, if for a given $\delta$ the approximation to the integral given by (2) is uniformly good. This fact, however, is already contained implicitly in the definition of the integrability.*

The theorem proved above is, however, not true if $f(x,\lambda)$ is merely integrable in the sense of Lebesgue. A further condition on the function must be imposed. As an example of such a condition we mention the following, that $f(x,\lambda)$ may be uniformly approximated to by a sequence of functions $f_i(x,\lambda)$, each of which is integrable with respect to $x$ (in the sense of Riemann) uniformly in $\lambda$, the approximation being uniform in the sense that there corresponds to any given $\epsilon > 0$, a constant $i_0$ independent of $\lambda$ such that

$$\int_a^b |f(x,\lambda) - f_i(x,\lambda)|\,dx < \epsilon \quad \text{for} \quad i \geqq i_0 .$$

In the important special case when $f$ depends only on $x$ this condition is always satisfied.

It should be noticed that the condition above does not suppose that the function $f(x,\lambda)$ is bounded.

DARTMOUTH COLLEGE

* Cf. Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig-Berlin, 1918, pp. 450–453.

# A THEOREM ON CONNECTED POINT SETS WHICH ARE CONNECTED IM KLEINEN*

BY R. L. WILDER

It has been shown by Miss Mullikin† that if $K$ and $M$ are two closed,‡ mutually exclusive point sets and $H$ is a closed, bounded, connected set having at least one point in common with each of the sets $K$ and $M$, then there exists a point set $L$, a subset of $H$, such that $L$ is connected and contains no point of either $K$ or $M$, but such that $K$ and $M$ each contain at least one limit point of $L$.

If $H$ is not closed, the above theorem no longer holds, as can be shown by very simple examples. It is the purpose of this note to establish an analogous theorem for the case where $H$, although not closed, is connected im kleinen.§

THEOREM. *Let $K$ and $M$ be two closed mutually exclusive point sets and $N$ a connected, connected im kleinen point set*

* Presented to the Society, October 31, 1925.

† *Certain theorems relating to plane connected point sets*, TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), pp. 144–162. Rosenthal gave a proof for that case where each of the sets $K$ and $M$ reduces to a single point. See A. Rosenthal, *Teilung der Ebene durch irreduzible Kontinua*, MÜNCHENER SITZUNGSBERICHTE, MATHEMATISCH-PHYSIKALISCHE KLASSE, 1919, p. 104.

‡ A point set is said to be (1) *closed*, if it contains all its limit points; (2) *connected*, if it is not the sum of two mutually exclusive points sets neither of which contains a limit point of the other; (3) *bounded*, if it lies entirely in a finite portion of the space under consideration.

§ A point set $M$ is said to be *connected im kleinen* at a point $P$ if for every circle $K_1$ with center at $P$ there exists a concentric circle $K_2$ such that every point $x$ of $M$ which lies interior to $K_2$ is joined to $P$ by a connected subset of $M$ which lies wholly within $K_1$. $M$ is itself said to be connected im kleinen if it is connected im kleinen at every point. See Hans Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, WIENER SITZUNGSBERICHTE, vol. 123, Abt. IIa (1914), pp. 2433–2489; also *Über die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist*, JAHRESBERICHT DER VEREINIGUNG, vol. 23 (1914), pp. 318–322. See also S. Mazurkiewicz, *Sur les lignes de Jordan*, FUNDAMENTA MATHEMATICAE, vol. 1 (1920), pp. 166–209, and earlier papers in Polish referred to therein.

*which has at least one point in common with each of the sets $K$ and $M$. Then there exists a point set $H$, a subset of $N$, such that $H$ is connected and contains no point of either $K$ or $M$, but such that $K$ and $M$ each contain at least one point of $N$ which is a limit point of $H$.*

PROOF. Let

$$N \times K = k,$$
$$N \times M = m.$$

For every point $P$ of $N-(k+m)$, consider the composant* $C(P)$ determined by $P$ in that set.

No $C(P) \equiv P$. For there exists a circle, $C$, with center at $P$, which encloses no point of the closed set $K+M$. As $N$ is connected and connected im kleinen, there exists at least one point $x$ of $N$, distinct from $P$, which lies, with $P$, in a connected subset of $N$ which lies wholly interior to $C$ and can, therefore, contain no point of $k+m$. Hence $x$ is a point of $C(P)$ distinct from $P$.

Every $C(P)$ has a limit point in $k$ or $m$. For suppose not. Then let $x$ be a point such that $C(x)$ has no limit point in $k+m$. Then

$$N = C(x) + N',$$

where $N'$ consists of all points of $N$ not contained in $C(x)$.

Either $C(x)$ contains a limit point of $N'$ or vice versa. Let $y$ be a limit point of $N'$ in $C(x)$. As $K+M$ is closed, there exists a circle $C$ with center at $y$ and enclosing no point of $K+M$. As $N$ is connected im kleinen there exists a point $z$ of $N'$ interior to $C$ which lies with $y$ in a connected subset of $N$ which lies wholly interior to $C$ and hence contains no point of $k+m$. Then $z$ is a point of $C(x)$. This is impossible. Hence $C(x)$ cannot contain any limit point of $N'$. Similarly $N'$ cannot contain any limit point of $C(x)$. Hence $N$ is the sum of two mutually exclusive sets neither of which contains a limit point of the other. But this is impossible

* I.e., the set of all points which lie with $P$ in a connected subset of $N-(k+m)$.

since $N$ is connected. Thus the supposition that a $C(P)$ exists which has no limit point in $k$ or $m$ leads to a contradiction.

If any $C(P)$ has limit points in both $k$ and $m$ the theorem is proved. Suppose this is not the case.

Separate $N$ into two sets, $N_1$ and $N_2$, such that

$$N_1 = k + Q_1 \quad \text{and} \quad N_2 = m + Q_2$$

where $Q_1$ contains all points $x$ of $N-(k+m)$ such that $C(x)$ has a limit point in $k$, and $Q_2$ contains all points $x$ of $N-(k+m)$ such that $C(x)$ has a limit point in $m$.

As $N$ is connected, $N_1$ contains a limit point of $N_2$, or vice versa. Suppose $N_2$ contains a limit point, $t$, of $N_1$.

The point $t$ cannot belong to $Q_2$, since in this case it could be shown that $t$ is joined to some point of $Q_1$ by a connected subset of $N-(k+m)$, by virtue of the connectedness im kleinen of $N$. Hence $t$ must be a point of $m$.

Let $T$ be a circle with center at $t$ and enclosing no point of $K$. There exists, because of the connectedness im kleinen of $N$, a connected subset, $R$, of $N$, which contains some point $s$ of $Q_1$ and $t$, and lies wholly interior to $T$. Let

$$[C(s)+R]-C(s)=R'.$$

The set $C(s)+R'$ is connected. Now $C(s)$ has no limit point in $R'$ since such a point would belong to the set $N-(k+m)$ and hence also to the set $C(s)$. On the other hand, if $R'$ has a limit point, $u$, in $C(s)$, it can be readily shown by application of the connectedness im kleinen of $N$ at $u$ that at least one point of $R'$ belongs to $C(s)$, which is of course impossible. Then $C(s)+R'$ is the sum of two mutually exclusive sets neither of which contains a limit point of the other, which contradicts the fact that $C(s)+R'$ is connected.

Thus the supposition that no $C(P)$ has a limit point in both $k$ and $m$ leads to a contradiction and the theorem is proved.

OHIO STATE UNIVERSITY

# A RAY OF NUMERICAL FUNCTIONS OF $r$ ARGUMENTS*

BY E. T. BELL

1. *Introduction.* Starting with any set $\Sigma$ of elements we may combine them according to rules having the formal properties (commutativity, associativity, distributivity) of algebraic addition ($A$), multiplication ($M$), subtraction ($S$), and division ($D$), provided the resulting combinations can be assigned self-consistent interpretations. Assuming this to be the case for a given $\Sigma$, we may then investigate the properties of systems $\Sigma'$ closed under one or more of $A$, $M$, $D$, $S$. It is not necessary to consider the properties of $\Sigma'$ with respect to those of $A$, $M$, $D$, $S$ that are omitted. There are thus conceivable precisely 15 systems $\Sigma'$. So far as systems $\Sigma'$ consisting of numbers (rational integers, rational numbers, algebraic numbers, algebraic integers) are concerned, it appears that 4 of the possible 15 have been deemed of sufficient interest to receive technical names. These are as follows: $AS$, module (Modul); $AMDS$, field (Körper); $AMS$, ring (Ring); $MD$, ray (Strahl), the last being due to Fueter.†

The elements of $\Sigma'$ need not be numbers to ensure interesting results, for example the algebra of classes and that of the relative product. Further significant theories have evolved from mere ova of $\Sigma'$'s; thus the theory of partitions is the $\Sigma'$ generated by a sort of parthenogenesis by $A$ alone from given rational integers. Doubtless with the continued evolution of arithmetic the neglected 11 will also be born, baptized, and investigated. Should this indeed come to pass it is fortunate that algebra has but 4, not 4000, fundamental operations.

* Presented to the Society, January 1, 1926.

† Introduced in his Dissertation; now current. Cf. *Synthetische Zahlentheorie* (Göschen's Lehrbücherei, 1921).

The ray isolates those properties of numbers and functions which are commonly called multiplicative.

The present note extends to $r>1$ arguments the ray devised in a former paper* to codify and extend the multiplicative properties of integers. The ray here relates to *sets* of $r$ integers, $r\geqq 1$. As in the case $r=1$ it was pointed out that the elements taken as arguments of the functions concerned could be identified with the elements of any abelian group, so here there is easily obtainable the obvious generalization to functions whose arguments are taken from $r$ abelian groups. Except in the special case when the numerical functions of $r>1$ arguments are products of $r$ functions each of a single argument, there is no longer, as in the case $r=1$, a second isomorphism with algebraic functions in $r$ variables. The properties of integers to which the case $r>1$ refers in general are genuine properties of a set of $r$ arbitrary constant integers, and they constitute an extension to such sets of the multiplicative properties of a single integer (rational or algebraic; if the latter, resolved into ideal factors). An instance of a function of $r>1$ arguments which is not essentially the product of $r$ functions of $r$ single arguments is the G. C. D. of $r$ integers.

The multiplication and division in the ray are specific operations having the abstract properties of $MD$, but not identical, in relation to the arguments, with the $MD$ of common arithmetic.

2. *Multiplication.* Call $f=f(x_1, \cdots, x_r)$ a numerical function of $x_1, \cdots, x_r$ if $f$ takes a single finite value when each $x_i$ is an integer $\neq 0$, and if further $f(1, 1, \cdots, 1)\neq 0$.

Let $N_r=(n_1, n_2, \cdots, n_r)$ be an ordered set of $r$ arbitrary constant integers each $>0$. Resolve $n_i$ into $t$ integral factors each $\geqq 1$,

$$(1) \qquad n_i=n_{1i}\, n_{2i} \cdots n_{ti}, \qquad (i=1, 2, \cdots, r).$$

Let $f_j(x_1, \cdots, x_r)$, $(j=1, 2, \cdots, t)$ be numerical functions. The $r$-fold sum

* This BULLETIN, vol. 28 (1922), pp. 111–122.

(2) $$\sum f_1(n_{11}, n_{12}, \cdots, n_{1r})\, f_2(n_{21}, n_{22}, \cdots, n_{2r}) \cdots f_t(n_{t1}, n_{t2}, \cdots, n_{tr}),$$

in which $\sum$ refers to all $n_{ji}(j=1, \cdots, t;\ i=1, \cdots, r)$ as defined which satisfy (1), will be written

(3) $$f_1 f_2 \cdots f_t$$

and will be called the product of $f_1, f_2, \cdots, f_t$ for the parameter $N_r$, or, when $N_r$ is understood, simply the product.

The number of terms in (3) when distributed as a sum of terms each of the form (2) is

$$\prod_{j=1}^{t} \nu(n_j)\{\nu(n_j)-1\} \cdots \{\nu(n_j)-t+1\},$$

where $\nu(n)$ = the number of divisors of $n$.

From (1), (2) it follows that the multiplication just defined is commutative and associative.

The unit of this multiplication is the numerical function $u=u(x_1, \cdots, x_r)$, which vanishes except when $x_j=1(j=1, \cdots, r)$, in which case $u=1$. By (3), if $f$ is any numerical function,

(4) $$uf=f.$$

In (4), as in every equality between numerical functions, it is understood that the functions have the same parameter; thus the parameter of $uf$ and of $f$ is $N_r$.

The zero of multiplication is the numerical function $w$ which vanishes for all parameters. Hence from (3),

(5) $$wf=w.$$

3. *Division.* Precisely as in the case of functions of a single argument* it can be shown that for $f$ any given

*TÔHOKU MATHEMATICAL JOURNAL, vol. 17 (1920), pp. 221–231; or more readily by an immediate extension of the method of generators for the algebras called $D$, $E$ in the TRANSACTIONS OF THIS SOCIETY, vol. 25 (1923), pp. 135–154. The generator of (3) is the ordinary product of $t$ "$r$-fold Dirichlet series," such a series being

$$Df \equiv \Sigma \frac{f(m_1, m_2, \cdots, m_r)}{m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r}},$$

where $\Sigma$ refers to $m_1, m_2, \cdots, m_r$ each ranging from 1 to $\infty$, and $s_1, s_2, \cdots$,

numerical function $\neq w$ there exists a unique $f'$, called the reciprocal of $f$, such that

$$ff' = u \,. \tag{6}$$

4. *The Ray* $\{f, g, h, \cdots\}$. From §§ 2, 3, it follows that the totality of numerical functions $f, g, h, \cdots$ of $r$ arguments form a ray when multiplication and division are as above defined.

5. *Examples.* As a first example we state and prove an extension of Dedekind's inversion ($r=1$) to $r>1$. Let $\zeta$ be the numerical function whose value is 1 for all values of the parameter, and let $\zeta'$ be the reciprocal of $\zeta$, so that $\zeta\zeta' = u$, the unit function. Then evidently

$$\zeta(n_1, n_2, \cdots, n_r) = \zeta(n_1)\zeta(n_2)\cdots\zeta(n_r) \,,$$

and the explicit definition of $\zeta'$ is easily seen to be as follows: $\zeta' = 0$ if any integer in the parameter is divisible by a square $>1$, otherwise $\zeta' = 1$ or $-1$ according as the number of those integers in the parameter which are divisible by an odd number of primes is even or odd, and hence

$$\zeta'(n_1, n_2, \cdots, n_r) = \zeta'(n_1)\zeta'(n_2)\cdots\zeta'(n_r) \,.$$

Let $f$ be any element of $\{f, g, h, \cdots\}$. Then by § 4, $\zeta f$ is in the ray and, by what precedes,

$$F = \zeta f \quad \text{implies} \quad f = \zeta' F \,, \tag{7}$$

the extension in question.

As a second example, let $a, b, f, g, h, k$ be elements of $\{f, g, h, \cdots\}$ between which there are the relations

$$af = bg \,, \qquad ah = bk \,. \tag{8}$$

Then, the indicated products being in the ray, we may eliminate $a, b$ as follows,

$$\frac{af}{ah} = \frac{bg}{bk} \,, \qquad \frac{f}{h} = \frac{g}{k} \,;$$

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$s_r$ are independent ordinary parameters. The generator of $u$ is $D_u = 1$; that of $f'$ in (6) is $1/D_f$. The generator of (3) is $\Pi D_f$, where $\Pi$ extends to $f_1, f_2, f_3, \cdots, f_i$. The condition $f(1, 1, \cdots, 1) \neq 0$ imposed in the definition in §2 is necessary to make the division of §3 always possible.

that is, (8) implies

$$fk = gh \,. \tag{9}$$

It may be of interest to state (8), (9) in ordinary notation: the equations

$$\sum a(n_{11}, \cdots, n_{1r}) f(n_{21}, \cdots, n_{2r}) = \sum b(n_{11}, \cdots, n_{1r}) g(n_{21}, \cdots, n_{2r}),$$

$$\sum a(n_{11}, \cdots, n_{1r}) h(n_{21}, \cdots, n_{2r}) = \sum b(n_{11}, \cdots, n_{1r}) g(n_{21}, \cdots, n_{2r}),$$

in which $\sum$ refers for $n_1, \cdots, n_r$ fixed to all positive integer solutions of

$$n_i = n_{1i} n_{2i} \qquad (i = 1, 2, \cdots, r) \,,$$

together imply, for the same $\sum$,

$$\sum f(n_{11}, \cdots, n_{1r}) k(n_{21}, \cdots, n_{2r}) = \sum g(n_{11}, \cdots, n_{1r}) h(n_{21}, \cdots, n_{2r}) \,.$$

For $r = 1$ this theorem is well known (due to Liouville).

With certain precautions and modifications (particularly as to the definition of zero elements) it is possible to construct for $f, g, h, \cdots$ a theory of each of the 15 possible types mentioned in the introduction. The ray however seems to be the only natural sequent to existing properties of numerical functions in the case $r = 1$.

THE UNIVERSITY OF WASHINGTON

# APPROXIMATE SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS BY LEAST SQUARES*

BY N. KRYLOFF

1. *Systems of First Order.* In many problems of mathematical physics and particularly in electrical circuit theory, it is of importance to find approximate solutions of a system of differential equations of the form

$$(1) \qquad \frac{dx_i}{dt} = f_i(t, x_1, x_2, \cdots, x_p), \qquad (i = 1, 2, \cdots, p).$$

Sometimes it may be shown by physical considerations that a system of type (1) which corresponds to some definite experimental fact, really possesses a periodic solution with a period equal to $T$. By a suitable change of variables, we may suppose that

$$(2) \qquad x_i = 0, \qquad (i = 1, \cdots, p) \qquad \text{for} \qquad t = 0, \quad t = T;$$

and it then remains only to find the numerical solution of (1) and (2) with a given degree of approximation. We shall suppose first that the system (1) is linear with variable coefficients, that is to say, that

$$(3) \qquad \frac{dx_i}{dt} - A_{i1}x_1 - A_{i2}x_2 - \cdots - A_{ip}x_p = F_i, \quad (i = 1, 2, \cdots, p),$$

where $A_{i1}, A_{i2}, \cdots, A_{ip}, F_i$ are functions of $t$. As in the method of least squares, we shall try to render stationary the integral

$$(4) \qquad \int_0^T \sum_{i=1}^{p} \left[ \frac{dx_i}{dt} - \sum_{k=1}^{p} A_{ik}x_k - F_i \right]^2 dt = \int_0^T \sum_{i=1}^{p} [L_i(x_i) - F_i]^2 dt$$

* Presented to the Society, December 29, 1925.

by the means of a sequence of the type

$$x_i = \sum_{k=1}^{n} a_{ik}^{(n)} \varphi_k(t) , \tag{5}$$

where the functions $\varphi_k(t)$ satisfy* the boundary conditions (2). For what follows, we must suppose also that the results of the modern theory of summation (for example, that of L. Féjer, or that of D. Jackson) are applicable to the system $[\varphi_k(t)]$. If this be true, it is possible to indicate the degree of approximation not only of

$$|x_i - X_i^{(n)}| , \tag{6}$$

but also of

$$\left| \frac{dx_i}{dt} - \frac{dX_i^{(n)}}{dt} \right| , \tag{7}$$

where $X_i^{(n)} = \sum_{k=1}^{n} b_{ik}^{(n)} \varphi_k(t)$ is a partial sum of order $n$ formed by means of Féjer's or Jackson's process, if $A_{ik}$, $F_i$ are to be supposed to satisfy the well known Lipschitz condition.

If we suppose that the coefficients $A_{ik}$, $F_i$ can be differentiated, the degrees of approximation of (6) and (7) are given by the recent investigations in the domain of the theory of functions of a real variable.† If we substitute (5) in (4) instead of $x_i$, and differentiate with respect to $a_{ik}$, it is easy to see that the conditions for a minimum take the form

$$\int_0^T \left\{ \left[ L_i(x_i^{(n)}) - F_i \right] \left[ \frac{d\varphi_k}{dt} - A_{ii}\varphi_k \right] + \sum_{r=1}^{r=p}{}^{(i)} \left[ L_r(x_i^{(n)}) - F_r \right] A_{ri}\varphi_k \right\} dt = 0 . \tag{8}$$

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* We may take, for example, $\varphi_k(t) = \sin kt$.

† See, e.g., de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Reélle*, Paris, Gauthier-Villars, 1919.

Multiplying the equations (8) respectively by $a_{ik}^{(n)}-b_{ik}^{(n)}$, adding them together, and assuming that the determinant of $A_{ik}$ is symmetric, we obtain

$$\int_0^T \sum_{i=1}^{p} [L_i(x_i^{(n)})-F_i][L_i(x_i^{(n)}-X_i^{(n)})]dt=0 .$$

Evidently we have also

$$\int_0^T \sum_{i=1}^{p} [L_i(x_i) - F_i][L_i(x_i^{(n)}-X_i^{(n)})]dt=0 ,$$

because $x_i$ are the solutions of (2); therefore

$$\int_0^T \sum_{i=1}^{p} L_i(x_i-x_i^{(n)})L_i(x_i^{(n)} -X_i^{(n)})dt=0 .$$

But we have

$$x_i^{(n)}-X_i^{(n)}=x_i^{(n)}-x_i+x_i-X_i^{(n)} ;$$

hence

$$\int_0^T \sum_{i=1}^{p}\left[ L_i(x_i-x_i^{(n)}) \right]^2 dt$$

$$= \int_0^T \left\{\sum_{i=1}^{p} L_i(x_i-x_i^{(n)})L_i(x_i-X_i^{(n)})\right\}dt . \tag{9}$$

If we utilise now the well known Bouniakowsky-Schwartz inequality, we find, from (9),

$$\int_0^T \sum_{i=1}^{p} [L_i(x_i-x_i^{(n)})]^2 dt \leqq \sum_{i=1}^{p} \sqrt{\int_0^T [L_i(x_i-x_i^{(n)})]^2 dt}$$

$$\times \sqrt{\int_0^T [L_i(x_i-X_i^{(n)})]^2 dt} .$$

Remembering what was said above about the approximation of $x_i$ and $dx_i/dt$, respectively, by $X_i^{(n)}$ and $dX_i^{(n)}/dt$, we obtain from (9)

$$\int_0^T [L_i(x_i-x_i^{(n)})]^2 dt < \epsilon_{in}, \qquad (i=1,2,\cdots,p), \tag{10}$$

where the order of magnitude of $\epsilon_{in}$ can be determined according to the restrictive conditions imposed on $A_{ik}$ and $F_i$. From (10), by the means of the same Schwartz inequality, we find immediately

$$\left| \int_0^x L_i(x_i - x_i^{(n)})dt \right| < \eta_{in}, \quad (i=1,2,\cdots,p),$$

where

$$\eta_{in} < \sqrt{\epsilon_{in}}.$$

Using the boundary conditions for $\varphi_k(t)$, we get the system of inequalities

$$\left| x_i - x_i^{(n)} + \int_0^x A_{ik}(x_k - x_k^{(n)})dt \right| < \eta_{in}.$$

The formulas for solving the system of integral equations of the second kind of the Volterra type enable us to state that

$$|x_i - x_i^{(n)}| < A\epsilon_{in}, \quad (i=1,2,3,\cdots,p)$$

where the order of magnitude of $\epsilon_{in}$ can be indicated and $A$ is a constant. Thus the following theorem is demonstrated.

THEOREM. *If the coefficients $A_{ik}$, $F_i(i, k=1, 2, \cdots, p)$ of the system* (3) *satisfy the Lipschitz conditions, and if the determinant formed from the $A_{ik}$ is symmetric then the effective calculation of the solutions of the system* (3) *can be derived from the method based upon the minimizing of the integral* (4). *Such a method, which may be called the method of least squares, gives not only a convergent process, but also the possibility of calculating the solutions of* (2) *with a given degree of approximation.*

2. *Systems of Second Order.* If instead of the system (3) we have to integrate the system of the differential equations of the second order

$$(11) \qquad \frac{d^2x_i}{dt^2} - A_{i1}x_1 - A_{i2}x_2 - \cdots - A_{ip}x_p = L_i(x_1x_2, \cdots, x_p) = F_i, \quad (i=1,2,3,\cdots,p),$$

with the boundary conditions $x_i(a)=x_i(b)=0$, then applying reasoning similar to that used before,* we can state that

$$\int_a^b [L_i(x_i-x_i^{(n)})]^2 dt < \epsilon_{in}\,, \qquad (i=1,2,\cdots,p), \tag{12}$$

where the order of smallness of $\epsilon_{in}$ can be fixed in advance corresponding to the supplementary restrictive conditions imposed on the coefficients $A_{ik}$, $F_i$. Then, if the determinant $A_{ik}$ is symmetric we can start from the obvious identity.

$$\int_a^b \{\sum_{i=1}^{p}\left[\frac{d(x_i-x_i^{(m)})}{dt}\right]^2+[A_{11}(x_1-x_1^{(m)})^2$$

$$+2A_{12}(x_1-x_1^{(m)})(x_2-x_2^{(m)})+\cdots]\}dt$$

$$=-\int_a^b \sum_{i=1}^{p} L_i(x_1-x_1^{(m)},\cdots,x_p-x_p^{(m)})(x_i-x_i^{(m)})dt\,.$$

If the quadratic form

$$A_{11}(x_1-x_1^{(m)})^2+2A_{12}(x_1-x_1^{(m)})(x_2-x_2^{(m)})+\cdots$$

is definitely positive, we easily obtain, by the use of the Bouniakowsky-Schwartz inequality,

$$|x_i-x_i^{(m)}|<\eta_m\,, \qquad (i=1,2,3,\cdots,p)$$

where the order of magnitude of $\eta_m$ can be fixed in advance according to the order of $\epsilon_{in}$.

In further communications, I shall state other applications of the method of least squares (and more generally of the method of least powers) which presents, it seems to me, a very powerful method for the approximate integration of differential equations of mathematical physics.

THE UNIVERSITY OF KIEFF

* In the case of one equation of the second order it was first stated in my recent communication to the French Academy. See N. Kryloff, *Sur une méthode, basée sur le principe du minimum pour l'intégration approchée des équations différentielles*, COMPTES RENDUS, vol. 181 (1925).

# NOTE ON A THEOREM OF KEMPNER CONCERNING TRANSCENDENTAL NUMBERS*

BY HENRY BLUMBERG

The theorem in question is the following one:†

*Let $a$ be an integer greater than* 1, *$p/q$ a rational fraction, $p \gtrless 0$, $q>0$; $\alpha_n (n=0, 1, 2, \cdots)$, any positive or negative integer smaller in absolute value than a fixed arbitrary number $M$, but only a finite number of the $\alpha_n$ equal to* 0; *then*

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{a^{2^n}} \left(\frac{p}{q}\right)^n$$

*is a transcendental number.*

Professor Kempner states: "The condition that only a finite number of coefficients shall be zero $\cdots$ I have not been able to remove."

Now although the proof of the theorem appears essentially to depend not merely on the *croissance* of the denominators $a^{2^n}$ but also on the particular character of the exponent $2^n$ of $a$, so that considerations based on the representation of numbers in the binary scale may be used, it nevertheless seems plausible that the restriction that only a finite number of coefficients shall be zero is dispensable. And, indeed, it is the purpose of this note to prove the theorem without this restriction; in other words, to prove the following theorem.

THEOREM. *The properly‡ infinite series*

$$f(x) = \sum_{n=0}^{\infty} \frac{\alpha_n}{a^{2^n}} x^n ,$$

*where $a$ is an integer greater than* 1, *and $\alpha_n$ an integer less in absolute value than a fixed number $M$, is transcendental for rational $x(\neq 0)$.*

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* Presented to the Society, April 22, 1916.

† TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 477.

‡ That is, the terms are not all zero after a certain point.

PROOF. Let $x=p/q$, where $p(\neq 0)$ and $q(>0)$ are integers. Suppressing the terms of $f(x)$ in which $\alpha_n=0$ we may write

$$f\left(\frac{p}{q}\right)=\sum_{n=1}^{\infty}\frac{\alpha_{\sigma_n}}{\alpha^{2^{\sigma_n}}}\,\frac{p^{\sigma_n}}{q^{\sigma_n}},$$

where $\alpha_{\sigma_n}\neq 0$ and $\sigma_n>\sigma_{n-1}(n=1, 2, \cdots)$. To prove that $f(p/q)$ is transcendental, we shall show that every polynomial

$$\phi(y)=A_0y^k+A_1y^{k-1}+\cdots+A_{k-1}y+A_k,$$

where $k\geqq 1$, $A_i(i=0, 1, \cdots, k)$ is an integer, and $A_0\neq 0$, is different from 0 for $y=f(p/q)$. We distinguish the two possibilities: $(a)$ for every $n$ there are 2 consecutive $\sigma_n$'s greater than $n$ and differing by more than $k$; $(b)$ after a certain point, the difference between 2 consecutive $\sigma_n$'s is less than or equal to $k$.

Case $(a)$. Let $n$ be such that $\sigma_n>\sigma_{n-1}+k$. We suppose that the individual terms of the expansion of $\phi(f(p/q))$ are retained without cancellation of common factors of numerators and denominators. Because of the rapid increase of the denominators absolute convergence is assured. Out of the various denominators of $\phi(f(p/q))$, we single out the following three:

$$d_1=a^{(k-1)2^{\sigma_{n-1}}+2^{\sigma_{n-2}}}q^{(k-1)\sigma_{n-1}+\sigma_{n-2}},$$

$$d_2=\left(a^{2^{\sigma_{n-1}}}q^{\sigma_{n-1}}\right)^k,$$

$$d_3=a^{2^{\sigma_n-\sigma_{n-1}}}q^{\sigma_n}.$$

We have:

$$\frac{d_3}{d_2}\geqq\frac{a^{2^{\sigma_n}(1-k/2^{\sigma_n-\sigma_{n-1}})}}{q^{k\sigma_{n-1}}}$$

$$>\frac{a^{2^{\sigma_n}(1-k/2^k)}}{q^{k\sigma_{n-1}}}$$

$$\geqq\frac{a^{2^{\sigma_n-1}}}{q^{k\sigma_{n-1}}}$$

and

$$\frac{d_2}{d_1}>a^{2^{\sigma_n-1}-2^{\sigma_{n-2}}}\geqq a^{2^{\sigma_n-1}-2^{\sigma_n-1}}=a^{2^{\sigma_n-1}-1}.$$

From the expressions for $d_1$, $d_2$, and $d_3$ and their ratios, it appears that for $n$ sufficiently large, $d_1$, $d_2$, and $d_3$ are consecutive if the denominators of $\phi(f(p/q))$ are arranged in ascending magnitude. The denominator $d_2$ occurs just once and with the numerator $n_2 = A_0\alpha_{\sigma_{n-1}}^k p^{k\sigma_{n-1}} \neq 0$; and

$$\left|\frac{n_2}{d_2}\right| < \frac{A_0 M^k |p|^{k\sigma_{n-1}}}{d_1 a^{2\sigma_{n-1}-1}}.$$

Hence

$$\frac{n_2}{d_2} = \frac{\epsilon_n}{d_1},$$

where $\epsilon_n \to 0$ as $n \to \infty$.

We shall now obtain an upper estimate of the absolute value of the sum $s$ of those terms of $\phi(f(p/q))$ whose denominators are $\geqq d_3$. A typical term $t$ of $\phi(f(p/q))$ has the form

$$A_{k-j}\frac{\alpha_{\sigma_{n_1}}\alpha_{\sigma_{n_2}} \cdots \alpha_{\sigma_{n_j}} p^{\sigma_{n_1}+\sigma_{n_2}+\cdots+\sigma_{n_j}}}{a^{2\sigma_{n_1}+2\sigma_{n_2}+\cdots+2\sigma_{n_j}} q^{\sigma_{n_1}+\sigma_{n_2}+\cdots+\sigma_{n_j}}};$$

a multinomial factor $m$ does not appear because, instead of regarding $mt$ as appearing once, we regard $t$ as appearing $m$ times.

The number $w_n$ of terms of $\phi(f(p/q))$ coming from the first $n$ terms of $f(p/q)$ is $n^k + n^{k-1} + \cdots + n + 1$—we are supposing that the terms are kept without combination as they arise initially in the expansion of $\phi(f(p/q))$, and we include terms that may possibly be zero because some $A_j = 0$; hence $w_n < n^{k+1}$. Hence surely the number of terms of $\phi(f(p/q))$ arising from the first $n$, but not from the first $(n-1)$ terms of $f(p/q)$ is less than $n^{k+1}$; moreover, any one such term is in absolute value less than $c|p|^{k\sigma_n}/a^{2\sigma_n}$, where $c$ represents the maximum absolute value of the numbers $A_{k-j}M^j$. Likewise, there are less than $(n+1)^{k+1}$ terms arising from the first $(n+1)$ terms, but not from the first $n$ terms of $\phi(f(p/q))$; and each of these terms is in absolute value less than $c|p|^{k\sigma_{n+1}}/a^{2\sigma_{n+1}}$. Therefore, since the terms of $\phi(f(p/q))$ with denominators $\geqq d_3$ involve the $n$th term or later terms of $f(p/q)$, we have

$$|s| < c\left[\frac{n^{k+1}|p|^{k\sigma_n}}{a^{2\sigma_n}} + \frac{(n+1)^{k+1}|p|^{k\sigma_{n+1}}}{a^{2\sigma_{n+1}}} + \cdots\right].$$

As may be seen through elementary considerations, the bracket is less than, say, double its first term, if $n$ is sufficiently large. For sufficiently large $n$, we therefore have

$$|s| < \frac{2cn^{k+1}|p|^{k\sigma_n}}{a^{2\sigma_n}} = \frac{2cn^{k+1}|p|^{k\sigma_n}q^{\sigma_n}}{d_3} < \frac{2cn^{k+1}|p|^{k\sigma_n}q^{\sigma_n+k\sigma_{n-1}}}{d_2a^{2\sigma_{n-1}}}.$$

Since $n \leqq \sigma_n$, the relative magnitude of $a^{2\sigma_{n-1}}$ renders the coefficient of $1/d_2$ infinitesimal for $n \to \infty$, so that

$$s = \frac{\epsilon_n'}{d_2},$$

where $\epsilon_n' \to 0$ as $n \to \infty$.

We may now see why $\phi(f(p/q)) \neq 0$. For the sum of those terms of $\phi(f(p/q))$ whose denominators are $\leqq d_1$ is either 0 or at least $1/d_1$ in absolute value. In the former case,

$$\phi(f(p/q)) = \frac{n_2}{d_2} + s = \frac{n_2 + \epsilon_n'}{d_2} \neq 0,$$

since $n_2$ is integral and $\neq 0$. And in the latter case,

$$|\phi(f(p/q))| \geqq \frac{1}{d_1} - \frac{|\epsilon_n|}{d_1} - \frac{|\epsilon_n'|}{d_2} < \frac{1 - |\epsilon_n|}{d_1} - \frac{|\epsilon_n'|}{d_1a^{2\sigma_{n-1}-1}} \neq 0,$$

for sufficiently large $n$, because $\epsilon_n$, $\epsilon_n'$, and $1/a^{2\sigma_{n-1}-1}$ approach 0 as $n \to \infty$.

Case ($b$). As in case ($a$), we set

$$d_3 = a^{2\sigma_n}q^{\sigma_n}.$$

Let $d_2$ equal the largest denominator in $\phi(f(p/q))$ less than $d_3$; and $d_1$, the largest denominator less than $d_2$; it is understood that denominators are not excluded because their corresponding numerators in the expansion of $\phi(f(p/q))$ happen to be zero on account of the vanishing of one or more of the $A$'s. The denominator $d_2$ is of the form $d_2 = \prod_{\nu=1}^{h} a^{2\sigma_{j_\nu}}q^{\sigma_{j_\nu}}$ where $h \leqq k$ and $\sigma_{j_\nu} < \sigma_n (\nu = 1, 2, \cdots, h)$. Moreover, $\sum_{\nu=1}^{h} 2\sigma_{j_\nu} < 2^{\sigma_n}$. For since $h \geqq 2$, or else (for large $n$), as is particularly evident from the sequel, there are denominators between $d_2$ and $d_3$, $\prod_{\nu=1}^{h} q^{\sigma_{j_\nu}} \geqq q^{2\sigma_n - k} > q^{\sigma_n}$ for $n$ sufficiently large, since (for large $n$)

$\sigma_{n-k} \geqq \sigma_n - k^2$. Consequently, if $\sum_{\nu=1}^{h} 2^{\sigma j_\nu} \geqq 2^{\sigma_n}$, it follows that $d_2 > d_3$. Since, then, $\sum_{\nu=1}^{h} 2^{\sigma j_\nu} < 2^{\sigma_n}$, we may conclude, if, as we supposed, there are to be no denominators between $d_2$ and $d_3$, that $h = k$ and that $\sigma_{n-1} \geqq \sigma_{j_\nu} \geqq \sigma_{n-k} (\nu = 1, 2, \cdots, k)$. (The inequality $\sum 2^{\sigma j_\nu} < 2^{\sigma_n}$ guarantees, for $n$ sufficiently large, that $\prod a^{2\sigma j_\nu} q^{\sigma j_\nu} < d_3$ even though $\prod q^{\sigma j_\nu} > q^{\sigma_n}$, as appears more clearly from the subsequent inequalities for $d_3/d_2$.) Hence we have, for large $n$,

$$\frac{d_3}{d_2} > \frac{a^{2\sigma_n} q^{\sigma_n}}{a^{2\sigma_{n-1}+2\sigma_{n-2}+\cdots+2^{\sigma_{n-k}}} q^{k\sigma_{n-1}}} > \frac{a^{2\sigma_n - k}}{q^{k\sigma_{n-1}}}.$$

From the definition of $d_1$, it follows that $d_1$, like $d_2$, is of the form $\prod_{\nu=1}^{k} a^{2\sigma j_\nu} q^{\sigma j_\nu}$ and, indeed, the same quantities $a^{2\sigma j_\nu} q^{\sigma j_\nu}$ occur, except that for one of these factors, say $a^{2\sigma j_\rho} q^{\sigma j_\rho}$ of $d_2$, in which $\sigma_{j_\rho} = \min\ \sigma_{\lambda_\nu}$, is substituted the factor $a^{2\sigma j_{\rho-1}} q^{\sigma j_{\rho-1}}$. Therefore

$$\frac{d_2}{d_1} \geqq \frac{a^{2\sigma_n - k} q^{\sigma_n - k}}{a^{2\sigma_{n-k-1}} q^{\sigma_{n-k-1}}} > a^{2\sigma_n - k - 1}.$$

The denominator $d_2$ is, except for the order of the factors $a^{2\sigma j_\nu} q^{\sigma j_\nu}$, obtainable in just one way; if, therefore, $n_2/d_2$ is the sum of the terms of $\phi(f(p/q))$ with denominator $d_2$, we have $n_2 = A_0^{\sigma} \prod_{\nu=1}^{k} \alpha_{\sigma j_\nu} p^{\sigma j_\nu}$, where the $\sigma_{j_\nu}$ are the same as in $d_2$, and $g$ is a multinomial coefficient in the expansion of a $k$th power. Hence $n_2 \neq 0$. Furthermore

$$\left| \frac{n_2}{d_2} \right| < \frac{A_0 g M^k |p|^{k\sigma_{n-1}}}{d_1 a^{2\sigma_{n-k-1}}} \leqq \frac{1}{d_1} \cdot \frac{A_0 g M^k |p|^{k\sigma_{n-1}}}{a^{2\sigma_{n-1}-k^2}}$$

for large $n$, since after a certain point, we have $\sigma_{\nu-1} \geqq \sigma_\nu - k$. Consequently

$$\frac{n_2}{d_2} = \frac{\eta_n}{d_1},$$

where $\eta_n \to 0$ as $n \to \infty$.

If, as in case $(a)$, $s$ represents the sum of the terms of $\phi(f(p/q))$ with denominators $\geqq d_3$, we have, as before, not-

withstanding the fact that denominators $\geqq d_3$ may now arise from the first $(n-1)$ terms of $f(p/q)$,

$$|s| < \frac{2cn^{k+1}|p|^{k\sigma_n}}{a^{2^{\sigma_n}}},$$

for large $n$. Hence

$$|s| < \frac{2cn^{k+1}|p|^{k\sigma_n}q^{\sigma_n}}{d_3} < \frac{2cn^{k+1}|p|^{k\sigma_n}q^{\sigma_n+k\sigma_{n-1}}}{d_2a^{2^{\sigma_n}-k}}.$$

Therefore

$$s = \frac{\eta_n'}{d_2},$$

where $\eta_n' \to 0$ as $n \to \infty$. From the fact that $n_2 \neq 0$, the equations $n_2/d_2 = \eta_n/d_1$ and $s = \eta_n'/d_2$, and the inequality for $d_2/d_1$, we conclude, as in case $(a)$, that $\phi(f(p/q)) \neq 0$.

As in the case of Professor Kempner's theorem, the exponent $2^n$ may be replaced by $b^n$, where $b$ is an integer $>2$. It is also obvious that $\alpha_n$ need not be limited, but it suffices, for instance, to subject it—for large $n$—to being $\leqq Ml^n$, where $M$ and $l$ are positive constants, and $l<2$.

THE OHIO STATE UNIVERSITY

# AN INVARIANT OF A GENERAL TRANSFORMATION OF SURFACES*

BY W. C. GRAUSTEIN

1. *Introduction.* If two surfaces, $S$ and $S'$, are in one-to-one point correspondence, the transformation $T$ of $S$ into $S'$ establishes between the pencils of tangent lines at corresponding points of $S$ and $S'$ a projective correspondence. Furthermore, if the line of intersection, $L$, of the tangent planes to $S$ and $S'$ at the corresponding points $M$ and $M'$ passes through neither of these points, that is, if neither $S$ nor $S'$ is a focal surface of the congruence of lines $MM'$, the pencils of tangent lines at $M$ and $M'$ cut $L$ in projective ranges of points.

The invariant cross ratio of the projectivity on $L$ is an invariant of the transformation $T$ with respect to the group of collineations of the three-dimensional space in which $S$ and $S'$ are imbedded. We propose to study this invariant, and to apply it, in particular, to the so-called fundamental transformations of surfaces.

2. *General Case.* We shall restrict ourselves primarily to the general case in which the projective correspondence on $L$ has two distinct fixed points, $D_1$ and $D_2$. Let the surfaces $S$ and $S'$ be represented parametrically so that corresponding points have the same curvilinear coordinates $(u,v)$. In particular, take as the $u$-curves the corresponding families of curves on $S$ and $S'$ whose tangents at corresponding points, $M$ and $M'$, intersect in $D_1$ and, as the $v$-curves, the curves whose tangents at corresponding points intersect in $D_2$.

A. *Fixed Points Finite.* If $D_1$ and $D_2$ are both finite points expressions for their coordinates, $y^{(1)} : (y_1^{(1)}, y_2^{(1)}, y_3^{(1)})$ and $y^{(2)} : (y_1^{(2)}, y_2^{(2)}, y_3^{(2)})$ are readily found. Since, for example,

* Presented to the Society, December 30, 1924.

$D_1$ lies on the tangent at $M$ to the $u$-curve on the surface $S$ which passes through $M$, there exists a scalar function $\lambda(u,v)$, such that $y^{(1)}=x+\lambda x_u$, where $x=x(u,v)$: $x_1(u,v)$, $x_2=x_2(u,v)$, $x_3=x_3(u,v)$ is the parametric representation of the surface $S$. Proceeding after this fashion, we obtain the equations

$$(1a) \qquad \begin{cases} y^{(1)}=x+\lambda x_u\,, & y^{(1)}=x'+\lambda' x'_u\,, \\ y^{(2)}=x+\mu x_v\,, & y^{(2)}=x'+\mu' x'_v\,. \end{cases}$$

Let the equations $u=u(t)$, $v=v(t)$ represent corresponding curves lying on $S$ and $S'$ and passing respectively through $M$ and $M'$, and denote by $P$ and $P'$ the points in which the tangents at $M$ and $M'$ to these curves intersect $L$. The coordinates $y$ and $y'$ of $P$ and $P'$ can evidently be written in the forms

$$y=x+\rho\frac{dx}{dt}\,, \qquad y'=x'+\rho'\frac{dx'}{dt}\,,$$

where $\rho(u,v)$ and $\rho'(u,v)$ are scalar functions.

Since $P$ is collinear with $D_1$ and $D_2$,

$$y=a_1y^{(1)}+a_2y^{(2)}, \qquad a_1+a_2\equiv 1.$$

The values of $a_1$ and $a_2$ are readily found to be

$$a_1=\frac{\rho}{\lambda}u'(t)\,, \qquad a_2=\frac{\rho}{\mu}v'(t).$$

Consequently, the ratio in which the point $P$ divides the line segment $D_1D_2$ is

$$\frac{a_2}{a_1}=\frac{\lambda}{\mu}\,\frac{v'(t)}{u'(t)}\,.$$

Similarly, the ratio in which $P'$ divides $D_1D_2$ is

$$\frac{a'_2}{a'_1}=\frac{\lambda'}{\mu'}\,\frac{v'(t)}{u'(t)}.$$

Hence, the invariant cross ratio of the projectivity on $L$, namely, the cross ratio $I$:

$$I=(D_1D_2PP')\,,$$

has the value

$$I=\frac{\lambda}{\mu}\Big/\frac{\lambda'}{\mu'}=\frac{\lambda\mu'}{\lambda'\mu}.$$

B. *Fixed Points at Infinity.* If $D_1$ and $D_2$ are both points at infinity, $L$ is a line at infinity and the tangent planes to $S$ and $S'$ at corresponding points are parallel: $S$ and $S'$ correspond by a parallel map.*

Since corresponding parametric curves have at corresponding points parallel tangents, scalar functions, $\lambda$, $\lambda'$, $\mu$, $\mu'$, exist so that

$$\text{(1}b\text{)}\qquad \begin{cases}\lambda x_u=\lambda' x'_u,\\ \mu x_v=\mu' x'_v.\end{cases}$$

By means of these equations it is readily shown that the invariant $I$ of the transformation of $S$ into $S'$ is given by the same formula as in the preceding case.

C. *One Fixed Point at Infinity, the Other Finite.* In this case corresponding tangent planes are not, in general, parallel but there exist two families of curves, one on each surface, which correspond and have at corresponding points parallel tangents. A map of this type we shall call *semi-parallel.*

If $D_1$ is the fixed point at infinity, it is the $u$-curves on $S$ and $S'$ whose corresponding tangents are parallel. Hence, we can write

$$\text{(1}c\text{)}\qquad \lambda x_u=\lambda' x'_u,\qquad y^{(2)}=x+\mu x_v,\qquad y^{(2)}=x'+\mu' x'_v.$$

Here, too, we find that $I$ is given by the same formula as before.

*In all three cases in which the projectivity on L has two distinct fixed points, the invariant I is given by the formula*

$$\text{(2)}\qquad I=\frac{\lambda\mu'}{\lambda'\mu}.$$

*For maps of this type the invariant $I$ was introduced and extensively employed in a previous paper by the author: *Parallel maps of surfaces*, TRANSACTIONS OF THIS SOCIETY, vol. 23 (1922), pp. 298–332.

3. *Associated Congruence.* Before leaving the case of two fixed points, it is worth while to note certain relationships between the transformation of $S$ into $S'$ and the congruence of lines joining corresponding points of $S$ and $S'$. It is geometrically evident that the focal planes of the congruence which pass through the line $MM'$ are the planes determined by $MM'$ and the fixed points, $D_1$ and $D_2$, of the correspondence on $L$. Hence, the developables of the congruence are the surfaces $v=$const. and $u=$const.

Let $F_1$ and $F_2$ be the focal points on $MM'$ which are associated respectively with the developables $v=$const. and $u=$const. The cross ratio, $(F_2F_1\,MM')$, of the four points on $MM'$ is equal, by a theorem of Chasles,* to the cross ratio of the tangent planes at these points to the ruled surface $u=u(t)$, $v=v(t)$ of the congruence. But these planes are precisely the planes through $MM'$ determined respectively by the four points $D_1$, $D_2$, $P$, $P'$ on $L$. Consequently,

$$I=(D_1D_2PP')=(F_2F_1MM') \,. \tag{3}$$

*The value of the invariant of the transformation of $S$ into $S'$ for an arbitrary pair of corresponding points, $M$ and $M'$, is equal to a cross ratio in which the points $M$, $M'$ are divided by the focal points of the associated congruence which lie on their line.*

Since, by equations (2) and (3),

$$(F_2F_1MM')=\frac{\lambda\mu'}{\lambda'\mu} \,,$$

it is reasonable to predict the following theorem.

*The ratios in which the focal points of the congruence which lie on the line $MM'$ divide the line-segment $MM'$ are $\lambda'/\lambda$ and $\mu'/\mu$.*

In fact, it is a simple matter to verify the fact that the coordinates, $x^{(1)}$ and $x^{(2)}$, of the focal points $F_1$ and $F_2$ are, in all three cases of § 2,

$$x^{(1)}=\frac{\lambda'x'-\lambda x}{\lambda'-\lambda} \,, \qquad x^{(2)}=\frac{\mu'x'-\mu x}{\mu'-\mu} \,.$$

* Journal de Mathématiques, (1), vol. 2 (1837), p. 413.

4. *Radial Transformations.* If the projective correspondence on $L$ is always the identity, every ruled surface of the congruence of lines $MM'$ is a developable, and conversely. But a characteristic of a congruence all of whose ruled surfaces are developables is that its lines all pass through a point, finite or at infinity.

*A necessary and sufficient condition that the projective correspondence on $L$ be always the identity is that the lines of the congruence joining corresponding points of $S$ and $S'$ all go through a point.*

A transformation of $S$ into $S'$ of this type we shall call a *radial* transformation. The invariant $I$ shall have the value unity.

5. *Fundamental Transformations.* If the developables of the congruence of lines $MM'$ cut the surfaces $S$ and $S'$ in conjugate nets, that is, if the parametric curves on both $S$ and $S'$ form conjugate nets, the transformation of $S$ into $S'$ is called a fundamental transformation or a transformation $F$.* For example, a parallel transformation is a transformation $F$. A semi-parallel transformation is not, in general.

Eisenhart's method† of establishing the general transformation $F$ of a surface $S$ into a second surface $S'$, or, better, of a conjugate net $N$ into a second (conjugate) net $N'$ is equivalent to expressing $F$ as the product of a parallel, a radial, and a second parallel transformation:

$$F: \qquad N \xrightarrow{P_1} \bar{N} \xrightarrow{R_2} \bar{N}' \xrightarrow{P_v} N',$$

where the intermediate transforms, $\bar{N}$ and $\bar{N}'$, of $N$ are themselves nets.

The transformation $P_1$ is an arbitrary parallel transformation

$$P_1: \qquad \bar{x}_u = hx_u, \qquad \bar{x}_v = lx_v.$$

* Cf. Jonas, SITZUNGSBERICHTE BERLINER MATHEMATISCHE GESELLSCHAFT, vol. 14 (1915), pp. 96–118; Eisenhart, TRANSACTIONS OF THIS SOCIETY, vol. 18 (1917), pp. 97–124, *Transformations of Surfaces*, Princeton University Press, 1923.

† *Transformations of Surfaces*, pp. 34–36.

If $\theta$ is a solution of the point equation of the given net $N$ and $\bar{\theta}$ is the "corresponding" solution of the point equation of the parallel net $\bar{N}$:

$$\bar{\theta}_u = h\theta_u , \quad \bar{\theta}_v = l\theta_v ,$$

the radial transformation $R_2$ is given by the equation

$$R_2 : \qquad \bar{x}' = \frac{\bar{x}}{\bar{\theta}} .$$

The second parallel transformation is, then,

$$P_v : \qquad x'_u = -\frac{\tau}{h}\bar{x}'_u , \qquad x'_v = -\frac{\sigma}{l}\bar{x}'_v ,$$

where

$$\sigma = \theta l - \bar{\theta} , \qquad \tau = \theta h - \bar{\theta} .$$

The invariants $I_1$, $I_2$, $I_3$ of the three transformations are, respectively,

$$I_1 = \frac{h}{l} , \qquad I_2 = 1 , \qquad I_3 = \frac{\tau}{\sigma}\frac{l}{h}.$$

Hence

$$I_1 I_2 I_3 = \frac{\tau}{\sigma} .$$

On the other hand, the transformation $F$, expressed directly, has the form

$$F : \qquad x' = x - \frac{\theta}{\bar{\theta}}\bar{x} ,$$

and the focal points $F_1$, $F_2$ of the congruence of lines $MM'$ have the coordinates

$$x^{(1)} = x - \frac{1}{h}\bar{x} , \qquad x^{(1)} = x - \frac{1}{l}\bar{x} .$$

Thus, the invariant $I$ of the transformation $F$ is found, by application of the first theorem of § 3, to have the value

$$(4) \qquad I = (F_2 F_1 M M') = \frac{\tau}{\sigma} .$$

*When the general transformation $F$ of a net $N$ is expressed in the Eisenhart form as the product of a parallel, a radial, and a second parallel transformation, the invariant of $F$ is equal to the product of the invariants of the individual transformations.*

Jonas also expresses the transformation $F$ as the product of three transformations *

$$F: \qquad N \xrightarrow{R_1} \bar{N} \xrightarrow{P_2} \bar{N}' \xrightarrow{R_3} N' ,$$

where

$$R_1: \qquad \bar{x} = \frac{x}{\theta} ,$$

$$P_2: \qquad \bar{x}'_u = \tau \bar{x}_u , \qquad \bar{x}'_v = \sigma \bar{x}_v ,$$

$$R_3: \qquad x' = -\frac{\theta}{\bar{\theta}} \bar{x}' .$$

Since

$$I_1 = 1 , \qquad I_2 = \frac{\tau}{\sigma} , \qquad I_3 = 1 ,$$

we have in this case, also,

$$I = I_1 I_2 I_3 .$$

*The invariant of the general transformation $F$ of a net is equal to the product of the invariants of the radial, parallel, and second radial transformations into which it can be factored.*

The theorem amounts to saying that the invariant of $F$ is equal to that of the parallel transformation $P_2$.

6. *Theorem of Permutability.* Outstanding in the theory of transformations $F$ is the theorem of permutability. According to this theorem, if a net $N$ is given and the nets $N'$ and $N''$ are two $F$ transforms of $N$ chosen at random, there exist $\infty^2$ nets $\bar{N}$, each of which is an $F$ transform of both $N'$ and $N''$.

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* Cf. Eisenhart, *Transformations of Surfaces*, pp. 40–46.

The transformation $T$ of the net $N$ into a chosen one of the nets $\bar{N}$ can be effected in either of two ways. For, since

$$N \xrightarrow{F_1'} N' \xrightarrow{F_2'} \bar{N}\,, \qquad N \xrightarrow{F_1''} N'' \xrightarrow{F_2''} \bar{N}\,,$$

we have

$$T = F_1' F_2' \quad \text{or} \quad T = F_1'' F_2''.$$

The invariants of the fundamental transformations $F_1'$, $F_2'$, $F_1''$, $F_2''$ are, by (4),

$$I_1' = \frac{\tau_1'}{\sigma_1'}\,, \qquad I_2' = \frac{\tau_2'}{\sigma_2'}\,, \qquad I_1'' = \frac{\tau_1''}{\sigma_1''}\,, \qquad I_2''' = \frac{\tau_2''}{\sigma_2''}\,.$$

But*

$$\tau_1'\tau_2' = \tau_1''\tau_2''\,, \qquad \sigma_1'\sigma_2' = \sigma_1''\sigma_2''\,.$$

Consequently,

$$I_1' I_2' = I_1'' I_2''\,.$$

*If a transformation of a net $N$ into a net $\bar{N}$ is equivalent to each of two pairs of transformations $F$, as described in the theorem of permutability, the product of the invariants of the transformations of the one pair is equal to the product of the invariants of those of the other pair.*

HARVARD UNIVERSITY

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* Cf. Eisenhart, *Transformations of Surfaces*, p. 50.

# WILCZYNSKI'S AND FUBINI'S CANONICAL SYSTEMS OF DIFFERENTIAL EQUATIONS*

BY E. P. LANE

1. *Introduction.* The projective differential geometry of surfaces has been studied extensively in the United States by Wilczynski, Green, and others, using the invariants and covariants of a completely integrable system of linear homogeneous partial differential equations. In Italy, much progress in projective differential geometry has been made by Fubini, Bompiani, and others, who have approached the subject from the point of view of differential forms and the absolute calculus. The *normal coordinates* of Fubini have been shown by him to be solutions of a canonical system of differential equations. It is the purpose of this note to show how Fubini's canonical differential equations may be obtained by Wilczynski's method, and to compare Wilczynski's and Fubini's canonical forms for the differential equations of a surface.

It is evidently desirable that geometers living on different sides of the Atlantic, writing in different languages, and using different analytic apparatus, but interested in the same subject, should be able to exchange ideas freely. It is hoped that this note will to some extent smooth the way for this commerce of ideas by showing how certain equations and formulas obtained in one notation may be written also in the other.

2. *Analytic Foundation.* Let the homogeneous coordinates $y^{(1)}, \cdots, y^{(4)}$ of a general point $P_y$ on a non-degenerate non-developable surface $S_y$ be given as analytic functions of two independent variables $u, v$. If $S_y$ is referred to its asymptotic

* Presented to the Society, April 2, 1926.

net, the four functions $y$ are solutions of a system of equations* of the form

$$(1)\qquad \begin{cases} y_{uu}+2ay_u+2by_v+cy=0\ , \\ y_{vv}+2a'y_u+2b'y_v+c'y=0\ . \end{cases}$$

The coefficients of these two equations are functions of $u$, $v$, and satisfy four integrability conditions, of which the only one that we shall use is

$$(2)\qquad a_v=b'_u.$$

This condition shows that there exists a function $p$ such that

$$(3)\qquad p_u=a,\qquad p_v=b'.$$

All transformations of the group

$$(4)\qquad y=\lambda(u,v)\bar{y},\quad \bar{u}=U(u),\quad \bar{v}=V(v)$$

are without geometric significance. In particular, the transformation $y=\lambda\bar{y}$ changes system (1) into another system of the same form whose coefficients are given by the following formulas:

$$(5)\qquad \begin{cases} \bar{a}=a+\dfrac{\lambda_u}{\lambda}\ ,\quad \bar{b}=b\ ,\quad \bar{c}=\dfrac{1}{\lambda}(\lambda_{uu}+2a\lambda_u+2b\lambda_v+c\lambda)\ , \\ \bar{a}'=a'\ ,\quad \bar{b}'=b'+\dfrac{\lambda_v}{\lambda}\ ,\quad \bar{c}'=\dfrac{1}{\lambda}(\lambda_{vv}+2a'\lambda_u+2b'\lambda_v+c'\lambda). \end{cases}$$

3. *Wilczynski's Canonical System.* Wilczynski observes that we shall have $\bar{a}=\bar{b}'=0$ if we choose $\lambda$ as a solution of the equations

$$(6)\qquad \frac{\lambda_u}{\lambda}+p_u=0\ ,\qquad \frac{\lambda_v}{\lambda}+p_v=0\ .$$

With this choice of $\lambda$ Wilczynski obtains for equations (1) the canonical form

$$(W)\qquad \begin{cases} y_{uu}+2by_v+fy=0\ , \\ y_{vv}+2a'y_u+gy=0\ , \end{cases}$$

* Wilczynski, *First memoir*, TRANSACTIONS OF THIS SOCIETY, vol. 8 (1907), pp. 233–260.

wherein $f$ and $g$ are defined by the formulas

$$(7) \qquad f=c-a_u-a^2-2bb',\ g=c'-b'_v-b'^2-2aa'.$$

*The canonical system* (W) *is characterized by the conditions* $a=b'=0$. The largest subgroup of the group of transformations (4) that preserves the form of system (W) is

$$(8) \qquad y=\text{const. } \bar{y}/\sqrt{U'V'}\ , \qquad \bar{u}=U(u)\ , \qquad \bar{v}=V(v)\ .$$

If the semi-covariant points $y$, $y_u$, $y_v$, $y_{uv}$ are used as vertices of a local tetrahedron of reference with a unit point suitably chosen, the local coordinates of a point $x$ on $S_y$ near $P_y$ are represented by the series*

$$(9) \qquad \begin{aligned} x_1&=1-\tfrac{1}{2}f\Delta u^2-\tfrac{1}{2}g\Delta v^2+\cdots\ ,\\ x_2&=\Delta u-a'\Delta v^2+\cdots\ ,\\ x_3&=\Delta v-b\Delta u^2+\cdots\ ,\\ x_4&=\qquad \Delta u\Delta v+\cdots\ . \end{aligned}$$

By means of these expansions the projective differential geometry of $S_y$ in the neighborhood of $P_y$ has been extensively studied.

4. *Fubini's Canonical System.* Referring to equations (5), let us observe that we shall have

$$(10) \qquad \frac{1}{2}\frac{(\bar{a}'\bar{b})_u}{\bar{a}'\bar{b}}+\bar{a}=0\ , \qquad \frac{1}{2}\frac{(\bar{a}'\bar{b})_v}{\bar{a}'\bar{b}}+\bar{b}'=0\ ,$$

if we choose for $\lambda$ a solution of the equations

$$(11) \qquad \frac{\lambda_u}{\lambda}+p_u+\frac{1}{2}\frac{(a'b)_u}{a'b}=0\ , \qquad \frac{\lambda_v}{\lambda}+p_v+\frac{1}{2}\frac{(a'b)_v}{a'b}=0\ .$$

With this choice of $\lambda$ we obtain for equations (1) the canonical form

$$(F) \qquad \begin{cases} y_{uu}-\dfrac{(a'b)_u}{a'b}y_u+2by_v+\varphi y=0\ ,\\[2ex] y_{vv}+2a'y_u-\dfrac{(a'b)_v}{a'b}y_v+\psi y=0\ , \end{cases}$$

* Wilczynski, *Second memoir*, TRANSACTIONS OF THIS SOCIETY, vol. 9 (1908), p. 98.

wherein $\varphi$ and $\psi$ are defined by the formulas

$$(12)\quad \begin{cases} \varphi = c - a_u - a^2 - 2bb' - b\dfrac{(a'b)_v}{a'b} - \dfrac{1}{2}\dfrac{\partial^2}{\partial u^2}\log(a'b) \\ \qquad\qquad + \dfrac{1}{4}\left[\dfrac{(a'b)_u}{a'b}\right]^2, \\ \psi = c' - b_v' - b'^2 - 2aa' - a'\dfrac{(a'b)_u}{a'b} - \dfrac{1}{2}\dfrac{\partial^2}{\partial v^2}\log(a'b) \\ \qquad\qquad + \dfrac{1}{4}\left[\dfrac{(a'b)_v}{a'b}\right]^2. \end{cases}$$

System (F) is equivalent to Fubini's canonical system. In fact Fubini shows* that his normal coordinates satisfy the equations

$$y_{11} = \beta y_2 + ny, \qquad y_{22} = \gamma y_1 + \nu y,$$

wherein subscripts indicate *covariant* differentiation with respect to the quadratic form $2\beta\gamma du\,dv$. In light of the well known formulas for covariant differentiation, we see that these equations are precisely the same as equations (F), provided that we place

$$(13)\qquad \beta = -2b,\ \gamma = -2a',\ n = -\varphi,\ \nu = -\psi\ .$$

*The canonical system* (F) *is characterized by the conditions*

$$(14)\qquad a = -\frac{1}{2}\frac{(a'b)_u}{a'b}, \qquad b' = -\frac{1}{2}\frac{(a'b)_v}{a'b}\ .$$

The largest subgroup of the group of transformations (4) that preserves the form of system (F) is

$$(15)\qquad y = c\bar{y}, \quad \bar{u} = U(u), \quad \bar{v} = V(v), \quad (c = \text{const.}).$$

The effect of the most general transformation (15) on the coefficients of system (F) is given by the equations

$$(16)\quad \bar{a}' = \frac{U'}{V'^2}a', \quad \bar{b} = \frac{V'}{U'^2}b, \quad \bar{\varphi} = \frac{1}{U'^2}\varphi, \quad \bar{\psi} = \frac{1}{V'^2}\psi\ .$$

* Fubini, *Fondamenti di geometria proiettivo-differenziale di una superficia*, Atti dei Lincei, (5), vol. 27 (1918), 2d semester, p. 45.

Therefore $\varphi$ and $\psi$, as well as $a'$ and $b$, are invariants. Moreover, it is evident that under this transformation the points $y_u, y_v, y_{uv}$ are covariant. A geometric description of these points and an interpretation of the invariants $\varphi$ and $\psi$ will be furnished in Section 6.

5. *Transformation from System* (W) *to System* (F). We shall now establish the transformation between the local tetrahedron with vertices at the points $y, y_u, y_v, y_{uv}$ for Wilczynski's canonical system (W) and the local tetrahedron with vertices at the points $y$, $y_u$, $y_v$, $y_{uv}$ for Fubini's canonical system (F).

In equations (9) let us replace $f$ and $g$ by the expressions given in equations (7). Then, for convenience, let us multiply each of the series thus obtained by a common proportionality factor, namely, the power series for the function $\Upsilon$ satisfying equations (6),

$$1 - a\Delta u - b'\Delta v + \tfrac{1}{2}(a^2 - a_u)\Delta u^2 + (ab' - a_v)\,\Delta u\Delta v + \tfrac{1}{2}(b'^2 - b'_v)\Delta v^2 + \cdots.$$

In this way we obtain the expansions

$$(17)\quad \begin{cases} x_1 = 1 - a\Delta u - b'\Delta v + (a^2 + bb' - \tfrac{1}{2}c)\Delta u^2 + (ab' - a_v\Delta u\Delta v) \\ \qquad + (b'^2 + aa' - \tfrac{1}{2}c')\Delta v^2 + \cdots, \\ x_2 = \quad \Delta u - a\Delta u^2 - b'\Delta u\Delta v - a'\Delta v^2 + \cdots, \\ x_3 = \quad \Delta v - b\Delta u^2 - a\Delta u\Delta v - b'\Delta v^2 + \cdots, \\ x_4 = \quad \Delta u\Delta v + \cdots. \end{cases}$$

These series represent the coordinates of a point $x$ on $S_y$ near $P_y$, in the notation of system (1), the tetrahedron of reference being the semi-covariant tetrahedron whose vertices are the points $y, \rho, \sigma, z$; the semi-covarients $\rho, \sigma, z$ being defined* by the equations

$$\rho = y_u + ax,\ \sigma = y_v + b'x,\ z = y_{uv} + b'y_u + ay_v + (ab' + a_v)y.$$

If a point has coordinates $x_1, \cdots, x_4$ referred to the tetrahedron $y, \rho, \sigma, z$ and has coordinates $\bar{x}_1, \cdots, \bar{x}_4$ referred to the tetrahedron $y, y_u, y_v, y_{uv}$ for system (1), then we have

$$x_1 + x_2\rho + x_3\sigma + x_4 z = k(\bar{x}_1 y + \bar{x}_2 y_u + \bar{x}_3 y_v + \bar{x}_4 y_{uv}),$$

* Wilczynski, *First memoir*, p. 248.

where $k$ is a proportionality factor. Thus we obtain the following equations of transformation between these two tetrahedrons:

$$(18)\qquad \begin{cases} \bar{x}_1 = x_1 + ax_2 + b'x_3 + (ab' + a_v)x_4\,, \\ \bar{x}_2 = \quad x_2 \quad + b'x_4\,, \\ \bar{x}_3 = \quad x_3 \quad + ax_4\,, \\ \bar{x}_4 = \quad x_4\,. \end{cases}$$

Using equations (17) and (18) we obtain then expansions for the coordinates of a point $x$ on $S_y$ near $P_y$, in the notation of system (1), the tetrahedron of reference being the tetrahedron $y$, $y_u$, $y_v$, $y_{uv}$ for system (1):

$$(19)\qquad \begin{cases} \bar{x}_1 = 1 - \tfrac{1}{2}c\Delta u^2 - \tfrac{1}{2}c'\Delta v^2 + \cdots\,, \\ \bar{x}_2 = \quad \Delta u - a\Delta u^2 - a'\Delta v^2 + \cdots\,, \\ \bar{x}_3 = \quad \Delta v - b\Delta u^2 - b'\Delta v^2 + \cdots\,, \\ \bar{x}_4 = \quad \Delta u\Delta v + \cdots\,. \end{cases}$$

If in equations (19) we substitute for $a$ and $b'$ the expressions given by equations (14) and replace $c$, $c'$ by $\varphi$, $\psi$ respectively, we obtain the coordinates of a point $x$ on $S_y$ near $P_y$ in the notation of system (F), the tetrahedron of reference being the tetrahedron $y$, $y_u$, $y_v$, $y_{uv}$ for system (F). Equations (13) enable us to write the same coordinates in the notation actually used by Fubini.

Therefore *it is possible to pass from an equation in local coordinates based on system* (W) *to the corresponding equations based on system* (F) *by using in succession equations* (7), (18), (14), *and substituting* $\varphi$ *for* $c$, $\psi$ *for* $c'$.

6. *Geometric Applications.* In the notation of system (W) the equation of the quadric of Lie, or osculating* quadric, is

$$x_1x_4 - x_2x_3 + 2a'bx_4^2 = 0\,.$$

By the method of the preceding section the equation of this

* Wilczynski, *Second memoir*, p. 82.

quadric in the notation of system (F) is found to be

$$(20)\quad x_1x_4-x_2x_3+\left[2a'b+\frac{1}{2}\frac{\partial^2}{\partial u\partial v}\log(a'b)\right]x_4^2=0\ .$$

Fubini, using system (F), has remarked* that *the line joining the two points $y$ and $y_{uv}$ is the projective normal.* Green, using system (W), has called† the line joining the points $y$ and $z$, where

$$z=y_{uv}+\frac{1}{2}\frac{(a'b)_v}{a'b}y_u+\frac{1}{2}\frac{(a'b)_v}{a'b}y_v\ ,$$

the *pseudo-normal.* These two lines are easily shown to be the same by means of the transformation established in the preceding section. In fact, the local coordinates of $P_z$ referred to the tetrahedron $y$, $y_u$, $y_v$, $y_{uv}$ for system (1) are found by equations (18) to be

$$\left(\ (\ )\ ,\ \frac{1}{2}\frac{(a'b)_v}{a'b}+b'\ ,\ \ \frac{1}{2}\frac{(a'b)_u}{a'b}+a\ ,\ \ 1\right).$$

Equations (14) show that, for system (F), this point is on the line joining the points $y$ and $y_{uv}$.

We are now in position to describe geometrically the points $y_u$, $y_v$, $y_{uv}$ for system (F). A simple calculation based on equation (20) shows that *the reciprocal polar of the projective normal, with respect to the quadric of Lie, crosses the tangent of the asymptotic $v=$*const. *at the point $y_u$ and crosses the tangent of the asymptotic $u=$*const. *at the point $y_v$.* Thus these points are characterized. In order to characterize the point $y_{uv}$, we observe that the locus of the point $y_u$, as $u,v$ vary, is a surface covariant to $S_y$. *The tangent at the point $y_u$ to the curve $u=$*const. *on the surface $y_u$ intersects the projective normal at the point $y_{uv}$, as does also the tangent at the point $y_v$ to the curve $v=$*const. *on the surface $y_v$.*

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* Fubini, *Alcuni risultati di geometria proiettivo-differenziale,* ATTI DEI LINCEI, (5), vol. 32 (1923), 2d semester, p. 324.

† Green, *Memoir on the general theory of surfaces,* TRANSACTIONS OF THIS SOCIETY, vol. 20 (1919), p. 126.

Inspection of system (F)reveals that *the tangent at the point* $y_u$ *to the curve* $v=$const. *on the surface* $y_u$ *is the line* $y_uy_v$ *if, and only if,* $\varphi=0$. Similarly, *the tangent at the point* $y_v$ *to the curve* $u=$const. *on the surface* $y_v$ *is the line* $y_uy_v$ *if, and only if,* $\psi=0$.

We shall now determine the developables and focal surfaces of the congruence of lines $y_uy_v$ reciprocal to the congruence of projective normals. Any point $Y$ on a line $y_uy_v$ (except the point $y_v$) is given by an expression of the form

$$Y=y_u+\lambda y_v .$$

As $P_y$ moves along a curve $v=v(u)$ on $S_y$, $P_y$ describes a curve. A point on the tangent of this curve is given by

$$Y'=Y_u+Y_v\frac{dv}{du}=\left(\lambda+\frac{dv}{du}\right)y_{uv}+\left(\phi+\lambda\frac{dv}{du}\psi\right)y+(\ )y_u+(\ )y_v.$$

If $P_y$ is a focal point of the line $y_uy_v$ and if the curve on $S_y$ corresponds to a developable of the congruence of these lines, then we have

$$\lambda+\frac{dv}{du}=0 , \qquad \varphi+\lambda\frac{dv}{du}\psi=0 .$$

Eliminating $\lambda$, we obtain the differential equation of the developables of the congruence,

$$\varphi du^2-\psi dv^2=0 . \tag{21}$$

And eliminating $dv/du$, we obtain

$$\psi\lambda^2-\varphi=0 . \tag{22}$$

Solution of this quadratic determines the focal surfaces of the congruence. The significance of the simultaneous vanishing of $\varphi$ and $\psi$ is thus made evident.

7. *Transformations from System* (F) *to System* (W). If we have an equation in local coordinates based on system (F) and wish to find the corresponding equation based on the sytem (W) we may proceed as follows. Replacing $\varphi$ and $\psi$ by the expressions given for them in equations (12), we obtain the corresponding equation in the notation of system (1), the

tetrahedron of reference being the covariant tetrahedron used with system (F). The vertices of this tetrahedron, besides $P_y$, are the points $\rho,\sigma,z$ expressed in the notation of system (1) by the formulas

$$(23)\quad \begin{cases} \rho = y_u + (a-\beta)y\,, \qquad \sigma = y_v + (b'-\alpha)y\,, \\ z = y_{uv} + (b'-\alpha)y_u + (a-\beta)y_v + [(a-\beta)(b'-\alpha) + a_v - \beta_v]y \end{cases}$$

wherein we have placed

$$(24)\qquad \alpha = -\frac{1}{2}\frac{(a'b)_v}{a'b}\,, \qquad \beta = -\frac{1}{2}\frac{(a'b)_u}{a'b}\,.$$

If a point has coordinates $x_1, \cdots, x_4$ referred to this tetrahedron $y,\rho,\sigma,z$ and has coordinates $\bar{x}_1, \cdots, \bar{x}_4$ referred to the tetrahedron $y,y_u,y_v,y_{uv}$ for system (1), we have the following equations of transformation:

$$(25)\quad \begin{cases} \bar{x}_1 = x_1 + (\alpha-\beta)x_2 + (b'-\alpha)x_3 + [(a-\beta)(b'-\alpha) + a_v - \beta_v]x_4\,, \\ \bar{x}_2 = \qquad x_2 \qquad\qquad + (b'-\alpha)x_4\,, \\ \bar{x}_3 = \qquad\qquad x_3 \qquad + (a-\beta)x_4\,, \\ \bar{x}_4 = \qquad\qquad\qquad x_4\,. \end{cases}$$

By means of this transformation we are able to find the equation of the given locus in the notation of system (1) and referred to the tetrahedron $y,y_u,y_v,y_{uv}$ for system (1). In order to obtain the desired equation, it only remains to place $a=b'=0$ and to replace $c,c'$ by $f,g$ respectively.

UNIVERSITY OF CHICAGO

# REPRESENTATION OF INTEGERS BY CERTAIN TERNARY CUBIC FORMS*

BY F. S. NOWLAN

1. *Introduction.* Apart from Eisenstein's theory relating to his canonical form† comparatively few results are available on representation of integers by ternary cubic forms. By methods differing from those employed by Eisenstein, this paper develops theory relating to two such forms. One form and the associated theory is, however, obtainable from Eisenstein's results. The other form cannot be so obtained.

2. *Rational Prime Factors of Norms of Integers of Cubic Number Fields.* Let $K(x)$ be the algebraic number field defined by a root $x$ of an irreducible cubic. Consider any integer $\pi$ of $K(x)$. Its norm, $N(\pi)$, is a rational integer. We seek the properties of its rational prime factors.

Let $N(\pi)=p_1^i p_2^j \cdots$, where $p_1, p_2, \cdots$ are rational primes and $i, j, \cdots$ are positive integers. Let $\pi_1$ be a prime factor of $\pi$. Then $\pi_1$ divides $N(\pi)$ and hence divides one of its rational prime factors, say $p_1$. It could not divide two such factors, for then it would divide their greatest common divisor and hence would be a unit.

Let us set $p_1=\pi_1\alpha$, where $\alpha$ is an integer of $K(x)$. Then $N(\pi_1)\cdot N(\alpha)=N(p_1)=p_1^3$. Four apparent possibilities arise:

$$N(\pi_1)=\pm 1, \qquad N(\alpha)=\pm p_1^3. \tag{1}$$

This must be excluded, as it would make $\pi_1$ a unit.

$$N(\pi_1)=\pm p_1, \qquad N(\alpha)=\pm p_1^2. \tag{2}$$

Then $\pi_1\pi_1'\pi_1''=\pm p_1=\pm\pi_1\alpha$. It follows that $\alpha=\pm\pi_1'\pi_1''$, where $\pi_1'$ and $\pi_1''$ are the conjugates of $\pi_1$. In this case $p_1$ is the norm of a prime of $K(x)$.

$$N(\pi_1)=\pm p_1^2, \qquad N(\alpha)=\pm p_1. \tag{3}$$

* Presented to the Society, December 29, 1925.

† JOURNAL FÜR MATHEMATIK, vol. 28, pp. 302–303.

Then $N(\pi_1) = \pm N(\alpha^2)$. Hence $\pi_1$ is associated with $\alpha^2$ and so is not a prime. Accordingly we exclude this case.

$$N(\pi_1) = \pm p_1^3, \qquad N(\alpha) = \pm 1. \tag{4}$$

Here $\alpha$ must be a unit and hence $p_1$ is a prime of $K(x)$. We therefore state

THEOREM I. *A rational integer, which may be represented by the norm of the general integer in a cubic number field $K(x)$, has its rational prime factors either* (a) *primes of the field, or else* (b) *norms of primes of the field. In case* (a) *these primes enter to powers which are multiples of three and in* (b), *such primes are themselves represented by the given norm.*

*3. Certain Properties of Two Galois Number Fields.* We consider the Galois fields defined by roots of the irreducible cyclic cubics

(i) $$x^3+x^2-2x-1=0,$$

and

(ii) $$x^3-3x+1=0.$$

For either equation we denote the roots by $x$, $x'$ and $x''$ and the corresponding cubic fields by $K_1(x)$ and $K_2(x)$, respectively. For both (i) and (ii), we have

$$x'=x^2-2\,, \qquad x''=x'^2-2\,, \qquad \text{and} \quad x=x''^2-2\,.$$

These relations arise* through giving a negative sign to the square root of the discriminant $\Delta$ in the formula

$$x'=\frac{1}{2\sqrt{\Delta}}\left[(6Q-2P^2)x^2-(9R-7PQ+2P^3+\sqrt{\Delta})x + (4Q^2-P^2Q-3PR-P\sqrt{\Delta})\right],$$

where $P$, $Q$, and $R$ relate to the general cubic $x^3+Px^2+Qx+R=0$.

PROPERTY I. *For both $K_1(x)$ and $K_2(x)$, integral algebraic numbers are given by*

$$A=\alpha_0+\alpha_1 x+\alpha_2 x^2$$

*with $\alpha_0$, $\alpha_1$, and $\alpha_2$ rational integers.*

* Serret, *Cours d'Algèbre Supérieure*, vol. 2, pp. 467–469.

PROPERTY II. *Unique factorization holds for the integers of both $K_1(x)$ and $K_2(x)$, since a sufficient condition in a cubic number field is that the class number of ideals shall be $h=1$.* This condition is satisfied for the two cubic fields.*

PROPERTY III. *The norm of the general integer $A=\alpha_0+\alpha_1 x+\alpha_2 x^2$ of $K_1(x)$ is given by the ternary cubic form*

$$(5)\quad F_1=\alpha_0^3-\alpha_0^2\alpha_1-5\alpha_0^2\alpha_2-2\alpha_0\alpha_1^2-\alpha_0\alpha_1\alpha_2+6\alpha_0\alpha_2^2-\alpha_1^3-\alpha_1^2\alpha_2-2\alpha_1\alpha_2^2+\alpha_2^3.$$

*The norm of the general integer of $K_2(x)$ is*

$$(6)\quad F_2=\alpha_0^3+6\alpha_0^2\alpha_2-3\alpha_0\alpha_1^2+3\alpha_0\alpha_1\alpha_2+9\alpha_0\alpha_2^2-\alpha_1^3+3\alpha_1\alpha_2^2+\alpha_2^3.$$

4. *Representation of Rational Primes as Norms of Integers of $K_1(x)$.* LEMMA (a). *The cubic form*

$$F=x^3+x^2y-2xy^2-y^3$$

*is congruent* (mod 7) *to* 0 *or to* $\pm 1$.

(b) *The congruence $x^3+x^2-2x-1\equiv 0 \pmod p$, with $p$ a rational prime, has solutions for $p=7$, or any prime of the form $7m\pm 1$.*

In proof of (a), we note that if $x\equiv y\equiv 0 \pmod 7$, then $F\equiv 0 \pmod 7$. If $y\equiv 0 \pmod 7$ and $x\not\equiv 0 \pmod 7$, then $F\equiv x^3\equiv \pm 1 \pmod 7$. If $y\not\equiv 0 \pmod 7$, we let $x\equiv zy \pmod 7$ and get $F\equiv y^3(z^3+z^2-2z-1)\equiv \pm(z^3+z^2-2z-1) \pmod 7$. On letting $z$ be congruent (mod 7) to $0, 1, \cdots, 6$, we obtain $F\equiv 0$ or $\pm 1 \pmod 7$.

Proof of (b) is given by Cailler, INTERMÉDIAIRE DES MATHÉMATICIENS, vol. 16 (1909), pp. 185–7.

We are now able to prove the following theorem.

THEOREM II. *For the field $K_1(x)$, defined by a root of $x^3+x^2-2x-1=0$, the norm of a prime not associated with a rational prime is either 7 or a rational prime of the form*

* L. W. Reid, *Tafel der Klassenanzahlen für kubische Zahlkörper*, pp. 60 and 74.

*$7m\pm1$. In case the prime is associated with a rational prime, the norm is the cube of the rational prime and is thus of the form $7m\pm1$. Rational primes of the forms $7m\pm2$ and $7m\pm3$ are primes of the field. Further, every rational prime $7m\pm1$ is factorable into three conjugate primes of the field and thus is the norm of a prime of the field.*

We note first that $N(1+x+x^2)=7$. Hence 7 is not a prime of $K_1(x)$, but may be represented in the form $F_1$ of (5). Let $\pi=\alpha_0+\alpha_1x+\alpha_2x^2$ be a prime of the field which is not associated with a rational prime. Then

$$\begin{aligned}N(\pi)&=\alpha_0^3-\alpha_0^2\alpha_1+5\alpha_0^2\alpha_2-2\alpha_0\alpha_1^2-\alpha_0\alpha_1\alpha_2+6\alpha_0\alpha_2^2+\alpha_1^3-\alpha_1^2\alpha_2\\&\qquad-2\alpha_1\alpha_2^2+\alpha_2^3\\&=(\alpha_0)^3+(-\alpha_1-2\alpha_2)(\alpha_0)^2-2(-\alpha_1-2\alpha_2)^2(\alpha_0)\\&\quad-(-\alpha_1-2\alpha_2)^3+7\alpha_2(\alpha_0^2+\alpha_0\alpha_1+2\alpha_0\alpha_2-\alpha_1^2-2\alpha_1\alpha_2-\alpha_2^2)\,.\end{aligned}$$

Hence $N(\pi)$ is congruent (mod 7) to an expression of the form*

$$x^3+x^2y-2xy^2-y^3,$$

which by the lemma is congruent (mod 7) either to 0 or to $\pm1$. Hence if $N(\pi)\neq7$, we must have $N(\pi)\equiv\pm1$ (mod 7) and thus $N(\pi)$ equals 7 or a rational prime of the form $7m\pm1$.

Rational primes $7m\pm2$ and $7m\pm3$ are primes of the field, for a rational prime $p$ is factorable into conjugate primes of $K_1(x)$ only when it is 7 or of the form $7m\pm1$.

To prove the last statement of the theorem, we note that

$$N(\alpha-x)=\alpha^3+\alpha^2-2\alpha-1.$$

Now let $p$ be a rational prime of the form $7m\pm1$ and suppose $p$ is a prime of the field. We have

$$\begin{aligned}(\alpha-x)(\alpha-x')(\alpha-x'')&=(\alpha-x)(\alpha+2-x^2)(\alpha-1+x+x^2)\\&=\alpha^3+\alpha^2-2\alpha-1\equiv0\ (\text{mod } p),\end{aligned}$$

* Writing $N(\pi)=(\alpha_0+2\alpha_1+4\alpha_2)^3-7(\alpha_0^2\alpha_1+\alpha_0^2\alpha_2+2\alpha_0\alpha_1^2+7\alpha_0\alpha_1\alpha_2+6\alpha_0\alpha_2^2+\alpha_1^3+7\alpha_1^2\alpha_2+14\alpha_1\alpha_2^2+9\alpha_2^3)$, we might dispense with Lemma (a). This method would, however, not apply to the form $F_2$.

by part (b) of the lemma. Hence $p$, a prime of $K_1(x)$, must divide one of the factors $\alpha-x$, $\alpha+2-x^2$ or $\alpha-1+x+x^2$. We would thus have a relation as $\alpha-x=p(\beta_0+\beta_1 x+\beta_2 x^2)$. In this case $p\beta_1=-1$, which is impossible since $p$ and $\beta_1$ are rational integers and $p>1$.

Hence no rational prime of the form $7m\pm1$ can be a prime of the field. But $p=7m\pm1$ divides $N(\alpha-x)$ and accordingly, by Theorem I, must be the norm of a prime of $K_1(x)$.

We may state the following corollary.

COROLLARY. *The form*

$$F_1=\alpha_0^3-\alpha_0^2\alpha_1+5\alpha_0^2\alpha_2-2\alpha_0\alpha_1^2-\alpha_0\alpha_1\alpha_2+6\alpha_0\alpha_2^2+\alpha_1^3-\alpha_1^2\alpha_2 \\ -2\alpha_1\alpha_2^2-\alpha_2^3$$

*represents* 1, 7, *all primes of the form* $7m\pm1$ *and all products whose prime factors are* 7 *and primes* $7m\pm1$. *It cannot represent primes* $7m\pm2$ *and* $7m\pm3$ *and has such primes as divisors only when they divide each of* $\alpha_0$, $\alpha_1$, *and* $\alpha_2$; *and then they appear in powers which are multiples of* 3.

5. *Representation of Rational Primes as Norms of Integers of* $K_2(x)$. Analogous to the lemma of §3, we have the following lemma.

LEMMA (a). *The cubic form*

$$F=x^3-3xy^2+y^3$$

*is congruent to* 0 (mod 3), *or else is congruent to* $\pm1$ (mod 9).

(b) *The congruence* $x^3-3x+1\equiv0$ (mod $p$), *with* $p$ *a rational prime, has solutions for* $p=3$, *or any prime of the form* $9m\pm1$.

Proof of (a) is obtained readily as in §3.

Proof of (b) follows from the proof for the corresponding statement with reference to $y^3-3y-1\equiv0$ (mod $p$), as given by L. Aubry in the INTERMÉDIAIRE DES MATHÉMATICIENS for November and December, 1924.

By analogy to Theorem 2, we have the following theorem.

THEOREM III. *For the field $K_2(x)$, defined by a root of $x^3-3x+1=0$, the norm of a prime not associated with a rational prime; is either 3 or a rational prime of the form $9m\pm1$. In case the prime is associated with a rational prime, the norm is the cube of the rational prime, and is thus of the form $9m\pm1$. Rational primes, other than 3 or those of the form $9m\pm1$ are primes of the field. Further, every rational prime $9m\pm1$ is factorable into three conjugate primes of the field and so is the norm of a prime of the field.*

In proof, we note first that

$$N(-1-x)=3.$$

Hence 3 is not a prime of $K_2(x)$, but may be represented in the form $F_2$ of (6). Now let $\pi=\alpha_0+\alpha_1x+\alpha_2x^2$ be a prime of the field which is not associated with a rational prime.

$$\begin{aligned}N(\pi)&=\alpha_0^3+6\alpha_0^2\alpha_2-3\alpha_0\alpha_1^2+3\alpha_0\alpha_1\alpha_2+9\alpha_0\alpha_2^2-\alpha_1^3+3\alpha_1\alpha_2^2+\alpha_2^3\\&=(\alpha_0+2\alpha_2)^3-3(\alpha_0+2\alpha_2)(-\alpha_1-\alpha_2)^2+(-\alpha_1-\alpha_2)^3\\&\quad+9\alpha_1\alpha_2(\alpha_0+\alpha_1+\alpha_2)\ .\end{aligned}$$

Thus $N(\pi)$ is congruent (mod 9) to an expression of the form

$$x^3-3xy^2+y^3,$$

which by the lemma is congruent to zero (mod 3), or else is congruent to $\pm1$ (mod 9). Hence if $N(\pi)\neq3$, we must have $N(\pi)\equiv\pm1$ (mod 9), and thus $N(\pi)$ equals 3 or a rational prime of the form $9m\pm1$.

Rational primes, other than 3 and those of the form $9m\pm1$, are primes of the field, for a rational prime $p$ is factorable into conjugate primes of $K_2(x)$ only when it is 3 or of the form $9m\pm1$.

Finally, we note that

$$N(\alpha-x)=\alpha^3-3\alpha+1.$$

Let $p$ be a rational prime of the form $9m\pm1$ and suppose $p$ is a prime of the field. We have

$$\begin{aligned}(\alpha-x)(\alpha-x')(\alpha-x'')&=(\alpha-x)(\alpha+2-x^2)(\alpha-2+x+x^2)\\&=\alpha^3-3\alpha+1\equiv0\ (\text{mod } p),\end{aligned}$$

by part (b) of the lemma. Hence $p$, a prime of $K_2(x)$, divides one of $\alpha-x$, $\alpha+2-x^2$, or $\alpha-2+x+x^2$. As in Section 3, this is readily seen to be impossible. Hence no rational prime of the form $9m\pm1$ can be a prime of the field. Whence by Theorem I, such primes are norms of primes of $K_2(x)$.

COROLLARY. *The form*

$$F_2=\alpha_0^3+6\alpha_0^2\alpha_2-3\alpha_0\alpha_1^2+3\alpha_0\alpha_1\alpha_2+9\alpha_0\alpha_2^2-\alpha_1^3+3\alpha_1\alpha_2^2+\alpha_2^3$$

*represents* 1, 3, *all primes of the form* $9m\pm1$ *and all products whose prime factors are* 3 *and primes* $9m\pm1$. *The form represents no prime other than these. Other primes are divisors of the form only when they divide each of* $\alpha_0$, $\alpha_1$, *and* $\alpha_2$, *and then they appear in powers which are multiples of* 3.

6. *Relation to Eisenstein's Canonical Form.** Eisenstein's canonical form for $p=7$, reduces to

$$u^3-21uv^2+21uvw-21uw^2+7v^3-105v^2w+84vw^2+7w^3.$$

This transforms into $F_1$ on the substitution

$$\begin{aligned} u&=\alpha_0-\frac{1}{3}\alpha_1+\frac{5}{3}\alpha_2\,,\\ v&=\qquad\quad\frac{1}{3}\alpha_2\\ w&=\quad\frac{1}{3}\alpha_1\,, \end{aligned}$$

with determinant equal to $-\frac{1}{9}$.

Thus the form $F_1$ and Theorem 2 might have been deduced from Eisenstein's results. The form $F_2$ is, however, not obtainable from Eisenstein's canonical form, since his form has relation to the division of the circle into $p$ parts, where $p$ is a prime of the form $3m+1$, whereas, the form $F_2$ has relation to the division of the circle into nine parts.

THE UNIVERSITY OF MANITOBA

* JOURNAL FÜR MATHEMATIK, vol. 28, p. 393.

## TONELLI ON CALCULUS OF VARIATIONS

*Fondamenti di Calcolo delle Variazioni.* By Leonida Tonelli. Bologna, Zanichelli. Vol. 1, 7+406 pp., 1921, 55 Lire; vol. 2, 8+660 pp., 1923, 80 Lire.

In its classical period, the calculus of variations depended for many of its pivotal theorems upon the theory of differential equations. This debt was partly offset by the contributions to the theory of boundary value problems made by Mason, Richardson, et al., in which existence and oscillation theorems were obtained, via the theory of integral equations, from the calculus of variations. But it is an entirely different procedure which underlies Tonelli's work. For the line integral whose extreme values are sought, is here studied by means of its direct functional dependence upon the curve along which it is taken. This is a new departure, at least in so far as its systematic use throughout the theory is concerned. Nothing of the sort is found in the texts based upon the Weierstrass theory (Kneser, Bolza). And there is not more than a suggestion of it in the first volume of Hadamard's *Leçons*.

By putting into the foreground the dependence of the integral upon the entire curve, the author focuses attention upon the "integral" aspects of the theory, rather than upon its "differential" characters. We get a calculus of variations which is concerned primarily with neighborhoods of curves, rather than with those of points. Another striking feature of the new theory, and one intimately connected with what has already been mentioned, is the preponderant importance of the theory of absolute extrema, which now assumes first place while the theory of relative extrema, which heretofore claimed the lion's share, is derived from it. There is possible, of course, a difference of opinion with regard to the fundamental significance of this new theory. Whether independence from the theory of differential equations is a desideratum may well be queried. It is very likely true that, for a long time to come at least, the classical theory will supply the most ready method of becoming acquainted with the calculus of variations and its problems. It must be admitted also that the importance of Tonelli's method would be greatly enchanced if it should prove possible to extend his theory to more general functionals than those which occur in the calculus of variations. On the other hand, the advances in the theory which the treatment of the integral as a function of lines makes possible seem to secure a permanent place for it.

The task of the reviewer has been considerably lightened by an article contributed by the author to volume 31 of this BULLETIN (p. 163), in which he explained how the functional conception of the calculus of varaitions led to the use of semi-continuity and related ideas which play a role of central importance in the theory. Volume I, for which its author was awarded the 1923 gold medal of the Italian Society of Sciences, is devoted

to a very careful and complete development of such preliminary material as his mode of treatment of the calculus of variations makes necessary. It contains, besides a historical sketch of the development of the subject, Parts I, II, III of 200, 145, and 120 pages respectively.

The first of these gives a useful treatment of some important concepts and methods which have come into use in the theory of functions of a real variable during recent years. Among them is a discussion of the Lebesgue integral which, in order to avoid implied acceptance of the Zermelo axiom, substitutes the concept of "pseudointerval" for that of "measurable set," and which follows closely the treatments of W. H. Young and de la Vallée-Poussin. In this connection we call attention to the further development of his ideas concerning the Lebesgue integral which Tonelli has given in a very interesting article in the ANNALI DI MATEMATICA, ser. 4, vol. 1 (1924). One finds here a sequence of theorems of Ascoli, Arzelà, Hilbert and Tonelli giving conditions which insure that an infinite set of functions or of curves possess a limiting element; the concepts of the "equivalence of a point set to an interval," of absolute continuity, of quasi-continuity, and the well known theorems on differentiability and other properties of the Lebesgue indefinite integral. The treatment excells in clarity of exposition and elegance of treatment, combining a wholesome regard for necessary details with emphasis upon essentials. It collects a great deal of material that is scattered in various journals and of which a good share is due to the author, presenting it in such a manner as to make it available to the student who has had a first course in the real variable.

After this introductory section we enter upon an alternating sequence maintained throughout both volumes, in which treatment of a certain problem in the parametric form is succeeded by the discussion of the same problem in the "ordinary" form (usually called the $x$-form). Part II entitled "Functions of Lines—Parametric Form" discusses the integral $\mathfrak{I}_C = \int_0^L F(x, y, dx/ds, dy/ds)ds$ as a function of the curve $C$ defined by by $x=x(s)$, $y=y(s)$, $0 \leqq s \leqq L$. The curves $C$ admitted are "ordinary curves," i.e., rectifiable continuous curves which lie entirely in a certain domain $A$ of the $XY$-plane; the function $F(x,y,x',y')$ is subjected to conditions of continuity, differentiability, and to the homogeneity condition. Of great convenience for the sequel is the classification of integrals $\mathfrak{I}_C$ in accordance with the sets for which the functions $F$ and $F_1$ vanish. This renders possible a systematic treatment of various cases which may arise and which heretofore have not always been clearly recognized. Thus, besides the regular and definite cases in which $F_1$ and $F$ respectively are different from zero for all points $(x,y)$ of $A$ and for all points $(x',y')$ for which $x'^2+y'^2 \neq 0$, there are considered the cases of quasi-regular, normally-quasi-regular, semi-normally-quasi-regular, and semi-definite problems characterized by restrictions on the loci of the zeros of the functions $F_1$ and $F$.

With these preparations out of the way, the author is now ready to attack the main problem of his first volume, viz., the determination of necessary and sufficient conditions for the semi-continuity of $\mathfrak{I}_C$. That continuity (defined in terms of "a neighborhood of a curve," as developed in

Part I), is too strong a condition to impose upon $\mathfrak{J}_C$ if the theory is to be at all applicable is made evident by the remark that the length integral $\int \sqrt{x'^2+y'^2}dt$ does not satisfy it, because it lacks upper semi-continuity; it is lower semi-continuous. In the next two chapters we find necessary and sufficient conditions for the semi-continuity of the function $\mathfrak{J}_C$ over the entire region $A$, or on a particular curve $C$. Much in these chapters consists of the author's own researches; there is in them a great deal to delight the reader who is sensitive to the beauties of a mathematical proof. Let me call attention to the proof that the condition $F_1(x,y,x',y') \geqq 0$ for every normalized pair $(x',y')$, throughout the set of points which are either interior to $A$ or are limiting points of such points, is necessary for lower semi-continuity of $\mathfrak{J}_C$; to the theorem that continuity of $\mathfrak{J}_C$ requires that $F \equiv P(x,y)x'+Q(x,y)y'$ at every point of the set just mentioned; to the theorem that if $\mathfrak{J}_C$ is positively definite and the region $A$ is of a certain character, then it is both necessary and sufficient for the lower semi-continuity of $\mathfrak{J}_C$ on an ordinary curve $C_0$ that the inequality $E(x_0,y_0;\cos\theta_0,\sin\theta_0;\cos\theta,\sin\theta) \geqq 0$ hold nearly everywhere on $C_0$. This last result, giving refreshingly novel significance to the Weierstrass condition, is of particular interest to anyone who knows what an important role this condition plays in the classical theory. Chapter 8, concerned with further properties of the integral $\mathfrak{J}_C$, begins with the theorem that if $C_0$ is an ordinary curve of length $L_0$ then every $e>0$ determines a neighborhood of $C_0$ and a number $d>0$ such that for every curve in this neighborhood, the inequality $|L-L_0|<d$ carries with it that $|\mathfrak{J}_C-\mathfrak{J}_{C_0}|<e$; this is followed by a second theorem to the effect that the second of the inequalities just mentioned implies the first, at least if $\mathfrak{J}_C$ is regular. Of this latter result a number of extensions are developed relaxing the conditions on $\mathfrak{J}_C$ and replacing $L$ by a general integral, running in close analogy to the theorems of Lindeberg. The remainder of the chapter is taken up by convergence and comparison theorems, of which the following is a sample: If $\mathfrak{J}_C$ is normally quasi-regular and if the sequence of ordinary curves $\{C_n\}$ converges uniformly to the ordinary curve $C_0$, then $\{\mathfrak{J}_{C_n}\} \to \mathfrak{J}_{C_0}$ carries with it that

$$\left\{\int_{C_n} G ds\right\} \to \int_{C_0} G ds,$$

if $G$ is positively homogeneous of degree 1 in $x'$ and $y'$, and continuous in $A$ and for every nonsingular pair $(x',y')$.

At the end of this chapter the reader has reached the climax of Volume I. The stage is set for the developments of the second volume. The only thing left is to treat the integral $I=\int f(x,y,y')dx$ as the integral $\mathfrak{J}_C$ was treated before; and this is done in Part III. While the results obtained here are in most instances entirely parallel to those of Part II, a considerable number of important changes are necessary in the proofs.

In Volume II, the calculus of variations proper begins. Part I (pp. 1–278) deals with the parametric case. The first question treated is that of the existence of absolute extremands among the elements of classes $K$ of ordinary curves $C$, i.e. the existence of ordinary curves $C$ of $K$ such that for every curve of $K$ we shall have $\mathfrak{J}_{C_0} \geqq \mathfrak{J}_C$ or $\mathfrak{J}_{C_0} \leqq \mathfrak{J}_C$. Two interesting

general criteria are followed by an array of twelve existence theorems for the integral in parametric form, which cover a wide group of cases, including all those which arise in the classical problems and applications, and in which semi-continuity plays an important role. The classes $K$ of curves relative to which the "absolute" extremand (the writer is aware here of a confrontation of opposites) is considered, are "complete classes" of ordinary curves, i.e. such as contain their limiting curves whenever these are rectifiable. The first of the existence theorems is a generalization of Hilbert's well known theorem; the subsequent theorems apply to integrals upon which various less restrictive conditions are imposed. Next follow extensions to the case in which the region $A$ is unlimited; and the existence of "extremands 'in piccolo'" closes the first chapter. In Chapter II, the author proceeds to discuss the properties of extremands. And it is here that we get for the first time a more intimate linking up with the classical theories. The cases of fixed end-points and of one or two variable end-points are treated in close parallelism. It is shown that the conditions of Legendre and of Weierstrass are necessary for an extremand $C_0$ and that along every arc of $C_0$ which possesses a continuously turning tangent, the Euler equations must be satisfied; an ordinary curve which has this property and which satisfies the Euler equations is called an *extremal*. It is shown that every ordinary minimizing curve must satisfy the equations

$$\int_0^s F_x ds - \frac{d}{ds}\int_0^s F_{x'} ds = C_1, \quad \int_0^s F_y ds - \frac{d}{ds}\int_0^s F_{y'} ds = C_2;$$

an ordinary curve satisfying these equations is called an *extremaloid*. The corner point condition, the transversality condition, the envelope theorems are derived in about the usual way. A much more detailed treatment than has hitherto been available is given of the conditions on the boundary of the domain $A$.

In Chapter III various existence theorems for extremals, their dependence upon the coordinates of the end-points, the continuity and differentiability of the functions representing them, all of which are usually obtained in consequence of extensions of theorems on differential equations are here proved directly by a method based upon the foundations laid in Chapter II concerning the existence of extremands and their relation to extremals. The last part of this chapter deals with broken extremals, here called "simple extremaloids," along the lines followed in the work of Carathéodory.

The following chapter is concerned with the theorems of Darboux and Osgood, and with Jacobi's condition. The latter is treated on the basis of a definition of "the focus of the point $P$ on the extremal $E_0$," which may be approximately rendered thus: the point $P_0$ is called the right-hand focus of $P$ on $E_0$, if it is the first point following $P$, such that for every $\rho$ there exists on $E_0$ or on its extension a point $P^*$ belonging to an extremal $E(P,P^*)$, not containing nor contained in the arc $E_0(P,P_0)$ and lying in a second-order $\rho$-neighborhood of $E_0(P,P_0)$. In this statement I have not specified what is meant by a "second-order $\rho$-neighborhood of a curve," because the general character of the meaning of this phrase is doubtless sufficiently clear from its use in other places, and because the statement as given should

suffice to indicate the characteristics of the author's mode of approach. The definition is followed by that of right and left-hand focus of a curve and then by the analytic determination of foci. The use of the word "focus" in such a way as to cover not merely the case of end-points variable along a curve but also that of fixed end-points, is becoming more and more general and has much to recommend it. The different cases, leading to the conditions of Jacobi, Kneser and Bliss, are treated in close proximity. They are followed by the condition of Poincaré for closed extremals and that of Carathéodory (as developed by Bolza) for broken extremals. There is also given, in small print, a derivation of Jacobi's condition by the classical method, based upon the properties of second-order linear differential equations and the transformation of the second variation. It is to be regretted that in the discussion of the second variation no mention is made of the elegant method which Bliss has used in recent years to obtain the Jacobi condition and the key idea of which lies in associating a new minimizing problem with the condition that the second variation be positive (see this BULLETIN, vol. 26 (1920), p. 343). The chapter ends with theorems concerning the uniqueness of the extremal joining two points.

Passing over the first four chapters of Part II (pp. 281–461) in which the ordinary case is treated we turn to Chapter IX, with which Part II closes and which takes up a number of the classical problems. One wonders upon approaching this chapter what effect the point of view of Tonelli is going to have upon the actual solution of problems, whether after all, when it comes to minimizing a particular definite integral, the old methods will not have to be used. To some extent this is indeed the case; there are, however, some interesting modifications in the manner of treatment. Take, for instance, the treatment of the classical problem of finding the surface of revolution of minimum area. The first fact established is that, in view of the general existence theorems, there always exists at least one minimizing curve $P_1P_2$ and that every arc of this curve which has no point in common with the axis of revolution (taken to be the $X$-axis) must be an extremal. It is then shown that if the minimizing curve has no points on the $X$-axis, it is an arc of catenary, if $P_1$ and $P_2$ have different abscissas, and a line parallel to the $Y$-axis if these abscissas are equal; also that, if the minimizing curve has at least one point on the $X$-axis, it consists of the broken line $P_1Q_1Q_2P_2$ formed by the perpendiculars from $P_1$ and $P_2$ to the $X$-axis, together with the segment of the $X$-axis between the projections $Q_1$ and $Q_2$ of these points. It follows that the solution is unique if $P_1$ and $P_2$ have the same abscissas and also if $P_1$ or $P_2$ are on the $X$-axis. For the case in which $P_1$ and $P_2$ are both above the $X$-axis and have distinct abscissas, it is shown that, if $P_2$ lies on a certain curve $S$, there are two curves furnishing an absolute minimum, one being an arc of catenary and the other the broken line $P_1Q_1Q_2P_2$. If $P_2$ lies in the one or the other of the two regions determined by $S$, there is a unique minimizing curve, in the one case the arc of catenary, in the other case the broken line. It is surprising that in the presentation of these results which are well known, although the author's method of obtaining them has some elements of novelty, no mention is made of the work of MacNeish and Sinclair, to whom doubtless

they are largely due. A similar omission is to be recorded with regard to Mason's work on the problem of the minimum moment of inertia. But to return to the treatment of the examples: While we shall still be obliged to solve differential equations for the discovery of extremals, the new method furnishes in a more direct way than was possible before, complete information concerning the existence of absolute extrema and the curves by which these are attained.

The third part of Volume II (pp. 465–567) deals with the isoperimetric problems, first in parametric form and then in the ordinary form. The method of treatment followed here is entirely analogous to that used in the earlier parts of the book. The existence theorems again take care of a very broad class of problems.

The fourth and final part (pp. 571–639) is concerned with the problem that usually receives first consideration, viz., that of the relative minimum. We find here the distinction between weak and strong extrema, sets of necessary and sufficient conditions for each of these, the introduction of the field concept, etc. Both the unconditioned problem and the isoperimetric problem are taken up in the parametric and in the ordinary forms, for fixed and for variable endpoints. That this can be done in so few pages is, of course, due to the fact that the necessary and sufficient conditions for the existence of absolute extrema can be turned to use by appropriate specification of the class $K$ of curves to which these conditions refer.

Little has to be added to this review. How useful the new method of treatment which Tonelli has introduced into the calculus of variations is going to be, will depend upon many things. Its value as a means of establishing the existence of absolute extrema for the cases thus far treated must be recognized; as a means of actually discovering extremizing curves, it has to rest upon the classical theory, which thus it extends rather than displaces. How far the theory can be extended so as to take care of the more complicated problems of the calculus of variations and so as to furnish a theory of maxima and minima in the functional calculus, only the future can tell. I do not at present see how it could be used for the purpose of initiating students into the subject; but, those who have been made familiar with the outstanding facts of the theory by the classical methods, and who have the proper acquaintance with the spirit and method of the real function theory should be able to use these books to great advantage and to find in them an ample field for study and research. I have already spoken of the clearness and elegance with which they are written. To this I only wish to add that the typography is excellent and that the very complete indexes and the systematic use of cross-references make it very convenient to read them. I am looking forward with great interest to the further volumes on this subject which the author has promised for the near future.

ARNOLD DRESDEN

## HISTORY OF FRENCH MATHEMATICS

*Histoire de la Nation Française. Tome* XIV, *Histoire des Sciences en France; première partie, Histoire des Mathématiques, de la Mécanique et de l'Astronomie.* Henri Andoyer and Pierre Humbert. Paris, 1924. xx+620 pp., of which 163 are devoted to the part under review.

This treatise is one of fifteen volumes on the history of the French nation from prehistoric times to the year 1920. The entire work is under the editorship of the distinguished scholar M. Gabriel Hanotaux of the Académie Française. This particular volume is edited by M. Émile Picard, a mathematician and general savant of very high rank, secrétaire perpétuel of the Académie des Sciences. The authors of the part devoted to the history of mathematics, mechanics, and astronomy are M. Andoyer, a member of the Académie des Sciences, and M. Pierre Humbert, professor in the university at Montpellier. Such personal mention is desirable as showing without argument the scholarly standing of those engaged in the work.

The dignity of the subject and the standing of those engaged in the undertaking are adequately paralleled by the dignity of the publication itself. Appearing in folio form, printed from type representing the modern trend toward a colorful page and release from eye strain, and upon paper of a dull finish, the book is reminiscent of the older format and the days of the best French presses.

As to the style in which the work is written there will naturally be various opinions, depending upon the bent of mind of the reader. The French are artists rather than mere mechanics, architects rather than mere structural engineers, poets rather than mere compilers of statistics. The work under review is frankly a work of literature as well as a scientific résumé; it has the charm that only the French language or the French training seems able to impart to a scientific study. Whether this is to be preferred to a lexiconic and laconic style is a matter that each reader must decide for himself. As for this reviewer, he envies it, and with an honest envy.

As to the scope and completeness of the work, it must not be expected that there can be adequately covered in 163 pages all the contributions made in more than a thousand years to the development of mathematics, mechanics, and astronomy in France. The entire volume would not suffice for this purpose if the aim were anything like completeness. What is set forth in essay form in the four chapters devoted to the subject may be summarized as follows:

Chapter I, from earliest time to Descartes,—only twenty-one pages, devoted chiefly to the middle ages and the renaissance.

Chapter II, pure mathematics from Descartes to Cauchy, with special attention to (1) the scientific academies and journals; (2) Descartes and

his period; (3) lesser scientists of the 17th century; (4) the rise of the calculus; (5) the middle of the 18th century; (6) the close of the century; — in all, fifty-eight pages.

Chapter III, mechanics and astronomy in the 17th and 18th centuries, forty-four pages, devoted chiefly to (1) statistics, dynamics, analytic mechanics; (2) the Copernican system in France, the problem of parallaxes, the figure and measure of the earth, celestial mechanics, and the metric system.

Chapter IV, from Cauchy to the present time, thirty-nine pages, devoted particularly to (1) the scientific schools and journals, (2) the theory of functions, (3) equations and groups, (4) differential equations, (5) geometry, (6) physics and mechanics, (7) astronomy and celestial mechanics, and (8) contemporary tendencies.

With such an extent of ground to be covered, any approach to completeness is impossible. The question that may therefore arise is as to whether the résumé is as complete as should be expected. In reply it will be observed that the authors not only give an interesting outline of French mathematical history, but freely go outside of their own country in the awarding of credit to such men as Isidorus, Cassiodorus, Boethius, and Jordanus Nemorarius, and also to such distinguished adopted sons as Alcuin, Adelard, Robertus Anglicus, Sacrobosco, and Roger Bacon. This is also done in the later period to men like the Bernoullis, Euler, Leibniz, Newton, Cayley, Sylvester, Gauss, Riemann, Georg Cantor, and Weierstrass.

The best résumés of achievements are perhaps found under the names of Viète, Descartes, Pascal, Desargues, Roberval, Laplace, Lagrange, Monge, Legendre, and Cauchy. The brief survey of present tendencies is also well written. The most valuable part of the work, from the mathematical standpoint, is naturally found in Chapter IV, which treats of the century from 1820 to 1920.

As to minor points, of adverse criticism, most of them unworthy of serious consideration: there is the mistake of a century in the case of Sylvester II (p. 8), and in the common repetition of the inaccurate statement that "La naissance du calcul des probabilités" occurred in the gambling problem proposed to Pascal and Fermat, when it had long before been discussed by Pacioli, Cardan, and Tartaglia. It seems, also, hardly worth while to ask whether the title of a book should be given precisely as in the edition named, as a bibliographer would desire, or given in the modern form. The question arises with numerous titles, such as Bachet's *Problemes plaisans et delectables*, (instead of *Problèmes plaisants et délectables*, as here given), and *La Geometrie* of Descartes (instead of the modern *La Géométrie*). By way of criticism, it is also proper to refer briefly to the illustrations of a historical work of this nature, particularly in a case like this in which there is a decided departure from the conventional style. The artist, M. Mestchersky, has attempted three things: (1) to give a goodly number of portraits; (2) to reproduce illustrations from classical treatises; and (3) to give in polychrome certain scenes from the past, mostly imaginative. His art is modern; he seeks to employ the strong type of wood engraving

which is seen in some of the early work of the fifteenth century; and his aim has been to bring out characteristics rather than finished details. Frankly, however, as to furnishing good likenesses of the men portrayed, the results are very unsatisfactory.

As a general summary, the work is a fair presentation of the claims of France, it is delightfully written, and it is worthy of careful reading by everyone of mathematical tastes. The study of the nineteenth century will be particularly helpful to the student; it is hardly intended as mathematics, but it is very well-written history.

DAVID EUGENE SMITH

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## SECRIST ON STATISTICAL METHODS

*An Introduction to Statistical Methods*, Revised Edition. By Horace Secrist. New York, The Macmillan Company, 1925. xxxiii+584 pp.

The most notable developments in mathematics have had their origins in attempts to solve difficult practical problems. The purpose of mathematics is not to make easy things hard but to make hard things easy. If it sometimes seems to the layman that the former is true, this is because, with the powerful tools at hand, the mathematician frequently undertakes difficult things. The association between mathematics and difficulty is so close that the illusion is created that there is a necessary connection, and that, to avoid difficulty, one must at all costs avoid mathematics. This illusion is rather common among the economic-business group of statisticians.

There is a large section of Secrist's book which contains no mathematics, and there is no objection to its exclusion from this portion, for here the ideas portrayed are so simple that they can be conveyed easily by pictures and numerical illustrations. This part, perhaps seventy-five per cent, is intended for the use of those who have not studied, or at least are not still accustomed to the use of high school algebra. So far as the reviewer is qualified to pass on this phase of the work, it appears to be a valuable contribution. It is fair to say that it does not possess distinct literary merit or marked individuality of approach. The author is rather over fond of quotation. Especially when some difficult point is to be explained, or a critical remark to be made, it is usually somebody else who is invoked to do it; so that at times it almost seems that we have a compendium of what various authorities have said, Fisher, Bowley, Mitchell, et al., and sometimes Pearson and Pearl, but nevertheless a very useful one, handy as a reference book—notably Chapter XVI on indexes—up to date, and easy to read.

The other and newer part of the book deals with more difficult ideas. Rightly again, in view of the type of student for which the book was written, only the simplest of these are presented; but now it would seem to have been better for the author to have demanded a minimum of mathematics, certainly some algebra, and probably some analytics. In

fact, at times he seems to be forced into presupposing these subjects, inconsistently with his general attitude. On page 425, for example, he assumes that the student understands what is meant by the mathematical "slope" of a line, but on page 427 he gives a detailed explanation (by an inconvenient method) of how one plots a line. In general, however, no real mathematics is presupposed, and the result is a laboriousness of explanation which can hardly produce in the mind of the reader more than a rough approximation to understanding.

This deliberate avoidance of mathematics is so common in statistical literature that it is worth while noting some illustrations of it here. It requires a good deal of argumentation, if one is unwilling to use algebra, to convince a student that a change of base can be made by forming proportions between certain kinds of index numbers, but not between certain other kinds. But the essential point is only this:

$$\frac{a}{b}+\frac{c}{d}\neq 2\cdot\frac{a+c}{b+d}\,, \quad \text{usually, but} \quad \frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}\,.$$

These algebraic statements surely have a meaning for the bright high school boy, and they are the crux of the whole matter; therefore to omit them is to becloud with argument what is really very simple and already well known. On page 279 there is an example of "some do's and don'ts." This one is a don't. Again, it might be stated in the absurdly simple form of the first inequality just given above, and then it might be illustrated. Thus one would not have to learn it; one would understand it, and that would be sufficient. But, instead, no explanation whatever is given; we are asked to learn that the method leads to an incorrect result, and, if we ask why, we are led to suppose that it just happens that way, or perhaps it is paradoxical.*

The author thus refuses a searching analysis to his readers. Also, there are occasional paragraphs where his own thought does not seem to have been sufficiently clear and thorough. One such paragraph is on page 373, ending "Hence, the reliability of a sample may be expressed in terms of its probable error." Another instance is his discussion of "significance." "It is said conventionally (p. 370) that if a certain result is three or more times as large as its probable error it is 'significant'." At best this is not a carefully stated definition. Suppose the coefficient of correlation between two characters in a population were truly zero. This definition would *imply* that it would not be possible to get a reliable estimate of this coefficient by sampling from the population. "What is meant by this expression? (p. 371)—"that the odds against the appearance" of the measurement in question, (p. 372), are about 1 in 22. This statement is not a correctly stated deduction; the odds against the appearance of this measurement are infinite; the odds are about 1 in 22 against the appearance of a measure-

* Expecially because there is a reference at this point to a paper by the author on *A statistical paradox*, which is in fact only an illustration of another rather simple algebraic inequality.

ment differing from the true value by as much as three times its probable error. On page 429 there is another definition of significance, also "what it has become conventional to say." Six times the probable error is now demanded. It is not explained why the conventional practice is now different, although the difference in meaning is very great: odds of 1:22 in the one case and 1:19,200 in the other. Secrist writes exactly as if he had never heard of a genuine probability distribution which was skew. To him "pure chance" seems to mean always a normal distribution (p. 399), otherwise there is "bias" and anything may happen. On page 376 we further read that the normal is the probability distribution approached in the measurement of natural and physical phenomena "which logic and belief (*sic*) prompt us to expect." We do not understand what is in his mind here.

The book is well nigh perfect in technique, quite free from typographical errors, well printed on good paper, and well bound. Compared with the earlier edition (1917) of which it is a revision, it is "enlarged, simplified, rearranged, and more fully illustrated. New chapters on *The Theory of Probability and Some Properties of the Normal Law of Error Distribution*, and on *The Treatment and Correlation of Time Series* have been added. Those relating to *The Principles of Index Number Making and Using*, and *American Index Numbers Described and Compared*, have been entirely recast." An appendix, with tables, has been added. The book contains a very great deal of useful information, gathered from reliable sources and well indexed.

B. H. CAMP

# SHORTER NOTICES

*Elementi del Calcolo delle Variazioni.* By G. Vivanti. Bibliotheca di matematiche superiori diretta da Roberto Marcolongo e Gaetano Scorza. Messina, Giuseppe Principato, 1923. 296 pp.

One of the most interesting features of this book is the large number of examples in the calculus of variations which are discussed throughout the text. Theoretical questions have not been neglected, but after treating them concisely the author hastens to elucidate his theory by applying it to numerous illustrations. He has followed the example set by Kneser in listing his problems by number at the end of the book, with indications of the pages on which they are considered. There are twenty-four in all. Most of them are well known, but one may be especially mentioned. It is an isoperimetric problem (No. 14, p. 93) whose extremals are straight lines, which has only solutions with corners, and whose conjugate points are determined by a simple geometric construction.

Mathematicians value highly the books on algebraic geometry which are devoted to the theory of special algebraic curves or surfaces. It has always seemed to me that for similar reasons a book devoted entirely to the elucidation of special problems of the calculus of variations, in which the theory of each problem would be developed consecutively as far as it is known, would be of great value. The text of Professor Vivanti would give important assistance in preparing such a treatise.

The book before us is divided into two principal parts, devoted, respectively, to the conditions arising from the first and second variations, and there is an appendix on functional calculus and its applications to the proof of the existence of absolute minima. The variety of problems of the calculus of variations treated is large, the more important ones being the simplest problem in the plane, parametric problems, problems whose solutions have corners or arcs in common with the boundary of a region, isoperimetric problems, problems with variable end-points, problems of the Lagrange type, and those involving double integrals. One chapter in each part is devoted to each of these cases. The simplest problem in the plane in non-parametric and parametric forms is treated with considerable completeness, but the conditions deduced for the other problems are for the most part necessary conditions. In a number of cases references are given to Bolza for sufficiency proofs, and in others the proofs are omitted or passed over somewhat hastily.

I should like to call attention here to the method which I have used for deducing the necessary condition of Jacobi from the second variation.* It is widely applicable and relatively simple. In parametric cases particularly

* TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 195; Bliss, *Calculus of Variations*, p. 161; Smith, TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 459; Larew, *ibid.*, vol. 20 (1919), p. 1.

it avoids the use of the transformation of Weierstrass which is exceedingly ingenious and interesting, but complicated and not easy to generalize for spaces of higher dimensions. The method used by Vivanti is the historically very important one which has come to us through a process of development from the ideas in Legendre's fundamental memoir on the second variation.

It seems to me that the theoretical material in this interesting book on the calculus of variations is presented in a somewhat less clear and systematic style than that which one finds in the other very important and useful treatises which we have had from the pen of Professor Vivanti. This is doubtless due to the author's justifiable desire to arrive as rapidly as possible at the results which are of significance for the interesting special problems which he considers. His conciseness has led to some inaccuracies, as in the reference on page 132 to Bolza for a part of the proof of the Euler-Lagranbe multiplier rule. The text seems to imply a theorem which Bolza has not proved and which I do not believe to be correct. Limitations of space probably forbade the expansion of the material in the appendix. It is at present a very interesting outline of a chapter in the calculus of variations which is still in process of development. Professor Vivanti there regards the integrals of the calculus of variations as special instances of functions of lines, and applies to them Volterra's definition of the derivative of such a function. In order to be effective for these integrals, however, the definition must be modified slightly. To secure the derivative limit, one must make the first and second derivatives of the variation $\eta(x)$ approach zero, as well as impose the Volterra requirement that the variation itself and the interval on which the variation is different from zero are infinitesimals.* I believe that the differential corresponding to Volterra's derivative, or the differential of Fréchet, for a function of a line, would prove to be more convenient than the derivative in this connection. The definition of a generalized integral on page 289 needs modification if it is to be applicable to the non-parametric integrals which are considered in the preceding paragraphs.

G. A. Bliss

*Maschinenbauliche Beispiele für Konstruktionsübungen zur darstellenden Geometrie.* By Theodor Schmid. Leipzig and Vienna, Franz Deuticke, 1925. 52 pp., 25 plates.

This is a collection of a number of very simple examples in mechanical drawing for the use of the young "technician," who "as a rule is not able to apply the instruction which he got in the usual courses in descriptive geometry to practical examples."

To judge from the practical examples represented on the 25 plates, the courses in descriptive geometry which do not enable a student to apply the knowledge gained to the representation of such simple objects must indeed be in a bad state. In the opinion of the reviewer very little descriptive geometry is needed to understand such plates.

Arnold Emch

* Proceedings of the National Academy of Science, vol. 1 (1915), p. 173.

*Nichteuklische Geometrie.* By Dr. Max Simon. Edited by Dr. Kuno Fladt. Leipzig, Teubner, 1925. XVIII+116 pp. 16 mo.

A careful writer always puts more pains into writing his preface, than any other part of his work, largely in the hope of presenting his wares attractively, and inducing the reader to go further. And in most cases it is lost labor, for the reader skips said preface. It would be a great mistake to do so in the case of the book before us, for Fladt's preface explains the nature of all that follows. The reader must understand, then, that this is largely a book of piety. The story is as follows: Simon, whose knowledge of elementary geometry was unparalleled, as all students of his report on the progress of that subject in the nineteenth century know, gave a course of lectures in 1905, and again in 1911 on non-euclidean geometry. He probably hoped to publish them in book form some day, but did not live to see this wish realized, and after his death in 1918 the material was handed to Fladt. The latter determined to publish it out of respect to his late teacher, and to stick as closely as possible to the spirit of the original. Simon, in a similar spirit of reverence always tried to follow as closely as possible the classical methods of the early writers, Gauss, Bolyai, and Lobachevsky. Hence the work was an act of piety on the part of both. But each had to modify his original plan. Simon could not leave the subject exactly where it had stopped a hundred years before, and Fladt, educated in the more critical modern school, felt constrained to introduce all of Hilbert's axioms, in order to have a really solid foundation. He shows unusual skill in reconciling what he takes from Simon, with what comes from Hilbert and Liebmann, but looked upon as a scientific work, rather than as a memorial, the book has many shortcomings.

To begin with, it is almost entirely given to hyperbolic geometry, to the exclusion of elliptic or spherical, the two being insufficiently distinguished any way. Then it is almost entirely "in plano"; the amount of solid geometry is negligible. This one-sidedness is painfully characteristic. The authors have apparently heard of none of the recent texts, which are not in German, not even of Somerville's compendious bibliography of non-euclidean geometry, or the earlier and less complete one of Halsted. The endeavor is to stick closely to the euclidean model, paying as little attention to the axioms as will "get by" in this critical age. The first five sections are devoted to preliminary discussions, and to what one might call "gossip" about parallels. Hilbert's axioms are introduced in §6 and a number of theorems having no reference to parallels are deduced. Then the axioms about parallels which distinguish euclidean, hyperbolic, elliptic, and spherical geometry are stated and compared; the possibility of a geometry where two lines intersect more than twice is ruled out in a foot note. Then much care is given to showing that if one of the parallel assumptions is true once, it is always true. A neater way of stating this is to say that if there exist one quadrilateral where three angles are right angles, then if in a single instance the fourth angle is less than, equal to, or greater than a right angle, just that will always be the case. The authors assume the Archimedean axiom (continuity) "So weit wie notwendig." From

this point on they devote themselves almost exclusively to hyperbolic plane geometry in general, and Lobachevsky parallels in particular. It is needful to show a number of things about parallels, notably that if $l$ is one of the parallels to $m$, through a point $P$, it is one of the parallels through every other point on itself. Much attention is given to the three types of pencils of lines, those through a point, those all parallel to one another, and those all perpendicular to a given line. The main object is to develop the formula for the parallel angle, and so the method for drawing parallels. The leading idea is taken from Engel, and consists in establishing a correspondence between a right triangle, and a quadrilateral with three right angles.

The hyperbolic circle first appears on page 62 and is defined as the locus of the reflection of a point in the lines of a pencil. When the pencil consists in concurrent lines this gives the proper circle, when it is a parallel pencil, this is the queer circle called a horocycle, when the pencil is made up of lines perpendicular to a given line this should give the equidistant curve, i.e. the locus of all points at the same distance as the given point from the given line. It will, however, only give one half of that locus. Next comes an elaborate study of the concept of area, and of trigonometry. Twice we find the functional equation

$$f(x)+f(y)=f(x+y).$$

We are told that it is "geometrically evident" that this function is continuous, hence the solution is $f(x)=rx$. But if we were content with those things which are "geometrically evident" we should not bother with non-euclidean geometry at all, for it is sufficiently evident geometrically that if Euclid's parallel axiom is not true, the amount of its falsity is not enough to disturb the equanimity of sensible people. It is not thus that real mathematics is made. The subject of three-dimensional geometry is covered in eight pages at the end. Of such interesting subjects as Clifford parallels, not a trace.

Such is the book. Interesting and clear, and impregnated with the spirit of the founders, but, at best, one of many, presenting no striking advances over others already available.

J. L. COOLIDGE

*The Principles of Thermodynamics.* By George Birtwistle, Fellow of Pembroke College, Cambridge. Cambridge University Press, 1925. 163 pp.

The book is a careful presentation of the subject to students who are presupposed to have only a general knowledge of the physical science. For this reason the relation of statistical mechanics to thermodynamics, and the connection between magnetism and temperature, for which a special knowledge of dynamical or physical theory is required, are not included. The third law of thermodynamics is likewise not considered. The work is prefaced in its first chapter by concise historical and descriptive statements relating to the theory of heat, temperature measurement, laws of Boyle, Charles and Avogadro, and the conception of a perfect gas. The

work really begins with the two laws of thermodynamics, Carnot's cycle, and Kelvin's temperature scale. This is followed by chapters relating to the dissipation of mechanical energy, thermodynamics of a fluid, change of state, equilibrium of systems, osmotic pressure, thermoelectric phenomena, gas theory, and radiation, as well as chapters bearing on the application in engineering, refrigeration, and the porous plug experiments.

The material covered is such that a chemist or a mathematician should possess and is of the kind a student of physics or engineering would welcome as the first ground to be covered for his more advanced study. As a book to be used by students in American colleges it would find a place for our Junior Honor students. It is well written and has enough of historical statements to connect the various phenomena treated to make it of interest to the general student properly qualified with a fair knowledge of physics and calculus. The book can be well adapted to use for the class of scientific workers for whom it is particularly written, namely, those whose work would require some knowledge, though not exhaustive, of the subject of thermodynamics. The author might well have included further applications of thermodynamics in physical chemistry.

A. F. KOVARIK

*Géométrie du Compas.* By A. Quemper de Lanascol, with a preface by Raoul Bricard. Paris, Blanchard, 1925. xx+406 pp. 24 fr.

Professor Bricard begins his preface with the striking sentence "Le cercle est prestigieux." The statement is a true one historically and it is equally true scientifically. In nature, in decoration, and in geometry, "the circle is fascinating," and M. Bricard appropriately calls attention to its perennial interest by referring to the work of our Professor Coolidge on the circle and the sphere as a "témoigne de la vitalité de ce culte." The esteem in which the figure was held in the philosophical schools of the Greeks is well known; the geometry of a single opening of the compasses, which occupied so much attention in the Middle Ages is less often considered; while Mascheroni's classical treatise, *Geometria del Compasso* (1797) seems never to have received the recognition that it deserved. Written by one who was at the same time a priest, poet, physicist, philosopher, and mathematician, it naturally showed something of the dilettante in its style and was lacking in mathematical succinctness, but nevertheless it was a work of much originality and of genuine merit. Attracting the notice of Napoleon, it was by him brought prominently to the attention of the French mathematicians of the early days of the empire, and was translated into their language by Carette (1798), thereafter attracting the attention of such eminent scholars as Delambre and Monge.

What M. Quemper de Lanascol has done is to take the work of Mascheroni as a foundation on which to build, to add to it a large number of other constructions that have appeared in various books and journals during the past century and a quarter, and to increase the offering further by many original problems. He has replaced the prolixity of Mascheroni by a succinctness of statement that is much more in harmony with modern tastes in matters mathematical. Indeed he comments with much justice

upon the style of his predecessor in this field by expressing the doubt that many readers ever progress beyond the mere statements of the latter's propositions.

Mascheroni seems to have sought to write a practical treatise, one that could be used by artisans. M. Quemper de Lanascol, however, gives us a work intended primarily for geometers, a treatise in pure mathematics and interesting chiefly to those who study the science for its own sake. As Professor Bricard remarks in his preface: "A mon avis, la Géométrie du Compas doit se défendre comme se défendent les neuf dixièmes des mathématiques pures: elle porte témoignage de l'ingéniosité humaine."

Aside from the mere constructions, ingenious as they often are, the reader will find in "Livre III" a treatment of the application of inversion to the geometry of the compasses, and also a note on the *Géométrie Elémentaire du Compas* which M. Cousinery published in 1851, shortly before his death.

DAVID EUGENE SMITH

*Der Gegenstand der Mathematik im Lichte ihrer Entwicklung.* By H. Wieleitner. Leipzig and Berlin, Teubner, 1925. 61 pp.

This little volume constitutes volume 50 of the well known *Mathematisch-Physikalische Bibliothek* which has been published since 1912 under the general editorial direction of W. Lietzmann and A. Witting. The subtitles are as follows: Algemeines über die Mathematik und ihre Entwicklung, Die Geometrie der Griechen, Die Algebra, Die moderne Geometrie, Die höhere Analysis, and Mathematik und Wirklichkeit. The booklet is very readable and contains brief accounts of various fundamental developments of mathematics, including some of the most recent ones. While historical questions receive relatively much attention the author aims also to elucidate a number of the most general mathematical principles by means of elementary illustrations, and thus to serve as a guide to those who desire to secure some insight into what is most important in the extensive field of mathematical developments.

The author begins by comparing mathematics with an extensive country which no single man is able to explore completely, and hence all are naturally glad to hear the reports of those who have explored various parts thereof. He then gives a brief sketch of the contributions made by various ancient peoples, stating in particular, on page 5, that we do not need to consider the contribution made by the Chinese notwithstanding the established fact that they influenced the development of mathematics in India, since only a few onesided orientalists believe now that the latter influenced materially the development of Greek mathematics. As regards the origin of our common number symbols, as well as in other respects, our author expresses the most modern views, and hence the booklet is well adapted to serve those who seek to keep in touch with new developments relating to elementary mathematics and to deepen their insight into the most popular aspects thereof.

G. A. MILLER

*Gli Elementi d'Euclide e la Critica Antica e Moderna.* By Federigo Enriques (col concorso di diversi collaboratori), Libri I–IV. Rome, Alberto Stock, 1925. 323 pp.

The national institute of the history of mathematical and physical sciences has undertaken the preparation of a critical and comprehensive history of a number of the more fundamental branches of the subject. The plan in the present work is to accept the Greek text of Heiberg, and to accompany the new Italian translation with the development of each idea, occasionally referring the reader to more detailed discussions, when the adequate treatment could not be given within the plan of the present book. A list of the more important translations and of commentaries is given, and constant references are made to them. Each definition, axiom, postulate, and theorem is taken up, and its relation to the critical thought of the various ages is given. The text of the translation appears in large type, double leaded lines, and provided with excellent figures. The comments and criticisms are not relegated to foot-notes, but follow immediately that part of the text to which they apply, but printed in different type, and names of authors are displayed. The present volume contains books one to four; each is treated by a different author, but in such a manner that the presentation is homogeneous and unified.

The press work and proof reading have been well done. Apart from a few endings of German adjectives in the foot-notes the only error noticed is the date of the letter from Gauss to Bolyai, p. 50, there given as March 6, 1882.

The German and English mathematical public are well provided with similar discussions in the works of Simon and of Heath. Three quarters of a century ago Italy had one of the ablest in that of Flauti, but apart from the recent able treatment of Book I by Vacca it is not so well represented by modern treatises. The mathematical literature of Italy and of the world is enriched by this contribution from an entirely competent source.

VIRGIL SNYDER

*Traité du Calcul des Probabilités et de ses Applications.* By Émile Borel. Paris, Gauthier-Villars, 1925.

Fascicule I, tome I. *Principes et Formules Classiques du Calcul des Probabilités.* Leçons professées par É. Borel, redigées par René Lagrange. 160 pp.

Fascicule III, tome II. *Mécanique Statistique Classique.* Leçons professées par E. Borel, redigées par Francis Perrin. 148 pp.

These two works are parts of a proposed treatise on the theory of probability that evidently aims to be the most complete work of its kind. The first was written by René Lagrange from lectures by Professor Borel. It covers that part of the theory of probability which has come to be considered as classical. It is divided into two fairly distinct parts. In the first part, starting in the usual way with definitions, the author takes up the topics of discontinuous, continuous, and denumerable probabilities. The inclusion of this last topic is one of the characteristics of this work

which distinguish it not only from works on probability by other authors but from Borel's earlier works on probability. Another characteristic is the introduction of the terms "problems of the first order," "problems of the second order." Problems of the first order are those concerned with the finding of a single probability, while those of the second order deal with the distribution of probabilities. The six classes of problems arising out of these two types of problems in each of the three kinds of probability form the basis of the arrangement of the subject matter. The second part of this work is devoted to a general discussion of the functions that arise in statistics with special emphasis on the theory of moments.

The second work was written by Francis Perrin from lectures by Borel. It is devoted to classical statistical mechanics and the kinetic theory of gases presented in much the usual arrangement. Borel takes considerable pains to show that the idea of probability can be introduced naturally into mechanics without making unreasonable hypotheses, and he spends considerable time on the bonds which unite the fundamental laws of kinetic theory with the principles of mechanics.

These two fasciculi are but a small part of what promises to be a great treatise on the theory of probabilities and its applications. While they intentionally stick to the older and more established topics they are written with a clarity and an elegance of detail and arrangement that prophesies much for the complete work. When finished there should be another great French work on probabilities to be placed besides the works of Laplace, Poisson, Bertrand, Poincaré, and the earlier work of Borel.

A. R. CRATHORNE

*Wie man einstens rechnete.* By E. Fettweis. Mathematisch-Physikalische Bibliothek, Band 49. Berlin, Teubner, 1923. 56 pp.

This small volume, issued for a quarter (one gold-mark) represents a fine type of popular scientific work common in Germany and Italy. There are chapters on the primitive origins of counting and reckoning, on Egyptian, Babylonian, Mayan and Chinese arithmetic, on computation among Greeks and Romans, on the abacists of the middle ages, on the arithmetic of the Hindus and of the Arabs, on the development of the Hindu-Arabic methods in the West, and on reckoning with counters.

The author is well acquainted with the modern literature of his subject. In the section on the Greek abacus, however, the author permits himself to enlarge upon what is actually known concerning the methods of computation with that instrument in ancient Greece. The methods of division and the like described by the author are found only in late times, about 1000 A.D. and thereafter, not in ancient times as stated here. In the treatment of Hindu mathematics, the methods of the early European works are included by Fettweis; we do not certainly know them to have been employed by the Hindus, although there is evidence showing that probability.

Any one interested in the history of arithmetic will find in this little volume a stimulating and entertaining account of the development of the subject.

L. C. KARPINSKI

*Conformal Representation.* By Leo Lewent. Translated by R. Jones and D. H. Williams. London, Methuen. viii+146 pp.

This book is one of the German series on science edited by Dr. E. Jahnke. It is a brief and, for the most part, elementary treatment of conformal representation in the theory of functions of a complex variable. It contains five chapters. The titles of the first three are I, The geometrical representation of complex quantities, II, The Cauchy-Riemann differential equation and conformal representation, III, Special transformations. The first two chapters, covering 22 pages, give the classical treatment of the subjects named in their titles. The third chapter contains treatments of the linear transformation, stereographic projection, inversion, Riemann surfaces, and detailed studies of the conformal representation of the interior of a circle on various special areas. This chapter seems to us the most valuable part of the book. We believe it might be of great use to the student beginning the study of functions of a complex variable. Chapter IV, Theorems and methods for the treatment of more general problems, covering 26 pages, is intended to give the student some insight into the more difficult problems of conformal representation. In our opinion, it does not well accomplish its purpose. The subject matter is not very clearly explained. The last chapter, V, Conformal representation of a circular area on the interior of a convex polygon, covering 25 pages, was written by W. Blaschke after Dr. Lewent's death. This chapter, intended as an application of the methods described in Chapter IV, is admirably written, better than any other part of the book.

The translation as a whole is satisfactory, though occasionally lapsing into unintelligible Germanisms. We think the book would be improved by an index. One fault of the book is that new terms introduced are frequently not defined but have attention drawn to them only by printing them in italics. We have noticed only a couple of typographical errors and those of little importance.

J. K. WHITTEMORE

*Synthetische Zahlentheorie.* By Rudolf Fueter. Göschens Lehrbücherei, Second revised edition. Berlin, Walter de Gruyter, 1925. viii+277 pp. Price 10 marks unbound, 12 marks bound.

Since this is a second edition of a favorably known book, published in 1917 and reviewed in the JAHRBUCH ÜBER DIE FORTSCHRITTE DER MATHEMATIK for that year, it seems necessary to note here only the changes made in the revision. These affect mainly the following three points: The part relating to the theory of exponential functions was entirely rewritten so as to include the entire theory as regards simply periodic functions, with a view to preparing the beginner for the theory of doubly periodic functions and complex multiplication; the present edition contains a complete existence proof of the fundamental units of a domain of rationality while the earlier edition assumed in this connection an important theorem proved in the well known work of Dirichlet-Dedekind; finally, there is found in the present edition a proof of the general theorem relating to cubic reciprocity, so that this case

is now treated here as completely as that relating to quadratic reciprocity. It is hoped that these changes and many slight improvements, involving together an increase of about six pages in the size of the book, will tend to increase the usefulness of this popular introduction to the modern theory of numbers.

G. A. MILLER

*Vorlesungen zur Einführung in die Relativitätstheorie.* By Ernst Richard Neumann. Jena, Fischer, 1922. viii+228 pp. M 4.50.

This volume is a very good exposition of the restricted and general theories of relativity, based upon lectures given by the author at Marburg in 1920. It contains an unfortunately brief (four page) appendix on Weyl's work, which prompts one to suggest that in a possible second edition it might be advisable to introduce a discussion of more recent progress in this field.

C. N. REYNOLDS, JR.

*La Relativité Générale, Théorie des Axes Mobiles.* By Gabriel Joly. Paris, Spes, 1925. 64 pp. 10 fr.

This pamphlet is a sequel to an earlier pamphlet by M. Joly entitled *Les Erreurs Philosophiques de M. Einstein, Etude Directe de la Relavitité* (reviewed by Professor A. A. Bennett in this BULLETIN, vol. 31, p. 567) in which he continues his attempt to reinterpret the theory of relativity while clinging to classical mechanics. It is a fair example of mechanical "fundamentalism."

C. N. REYNOLDS, JR.

*Die Determinanten.* By E. Netto. Leipzig, Teubner. 2d Edition. vi+122 pp. 1925.

This second edition differs little from the first (reviewed in this BULLETIN, (2), vol. 17 (1910–11), p. 547). The changes in text are minor save that the twelfth chapter of the first edition, on *Functional determinants*, has been omitted in this edition. This is no improvement. The typography is not as clear as in the first edition. It is quite misleading to put on the title page "verbesserte Auflage."

J. B. SHAW

# NOTES

The opening number of volume 28 of the TRANSACTIONS OF THIS SOCIETY (January, 1926) contains the following papers: *Intersections and transformations of complexes and manifolds*, by S. Lefschetz; *Divergent double sequences and series*, by G. M. Robison; *On certain families of orbits with arbitrary masses in the problem of three bodies*, by F. H. Murray; *On certain families of orbits with arbitrary masses in the problem of three bodies (second paper)*, by F. H. Murray; *Existence theorems for a linear partial difference equation of the intermediate type*, by C. R. Adams; *An algebra of sequences of functions with an application to the Bernoullian functions*, by E. T. Bell; *Bundles and pencils of nets on a surface*, by E. P. Lane; *On the momental constants of a summable function*, by R. E. Langer; *Fundamental systems of formal modular protomorphs of binary forms*, by W. L. G. Williams; *Non-synchronized relative invariant integrals*, by K. P. Williams.

The Verein Deutscher Ingenieure is preparing a general bibliography of all published or unpublished tables of functions, of importance in pure mathematics, physics, astronomy, or allied sciences. It asks that information about all such tables, completed or in preparation, be sent to the Verein at Berlin.

At its meeting in April, 1926, the National Academy of Sciences elected as foreign associates Professor Jacques Hadamard, of the Ecole Polytechnique and the Sorbonne, and Professor Max Planck, director of the Institute of Theoretical Physics of the University of Berlin. No new members were elected. Professor Oswald Veblen was elected to the Council of the Academy for a period of three years.

An extraordinary session of the General Assembly of the International Research Council was held in Brussels on June 29, for the consideration, among other matters, of proposed amendments to the statutes of the Council. The session was attended by official representatives of the principal scientific bodies of twenty of the countries affiliated with the Council. The American delegates were Dr. Vernon Kellogg, permanent Secretary of the National Research Council, Dr. F. G. Cottrell, Director of the Fixed Nitrogen Research Laboratory, and Professor G. D. Birkhoff, President of the American Mathematical Society. Unanimous action was taken inviting Germany, Hungary, Austria, and Bulgaria to join the Council.

A colloquium will be held at St. Andrews, Scotland, August 3–13, 1926, under the auspices of the Edinburgh Mathematical Society. Among the speakers will be Professor G. D. Birkhoff, who will deliver a course of lectures on *The significance of dynamics for scientific thought.*

The Brussels Academy announces the following subject for its prize for 1927: *A contribution to the solution of problems with axial symmetry in the theory of general relativity.* Competing memoirs should reach the Academy by July 31, 1927.

The decennial prize of the Belgian Government for the period 1914–23 has been awarded through the Brussels Academy to Professor C. J. de la Vallée Poussin, for his work in mathematical analysis.

The Franklin Institute has awarded its Franklin Medal and certificate of honorary membership to Professor Niels Bohr, of the University of Copenhagen.

The Russian Academy of Sciences has elected the following honorary members: Professor A. S. Arrhenius, of the University of Stockholm, Professor H. A. Lorentz, of the University of Leyden, Professor Max Planck, of the University of Berlin, and Professor Vito Volterra, of the the University of Rome. Professor Arnold Sommerfeld has been elected a corresponding member.

Professor G. Mittag-Leffler, of the University of Stockholm, has been elected a foreign member of the Göttingen Society of Sciences.

Professor L. Bieberbach, of the University of Berlin, has been elected an honorary member of the Calcutta Mathematical Society.

Professor Leon Lichtenstein, of the University of Leipzig, has been elected a member of the Saxon Academy of Sciences.

The Karlsruhe Technical School has conferred honorary doctorates on Professor Emil Müller, of the Vienna Technical School, and Professor F. Schur, of the University of Breslau.

The University of Hamburg has conferred an honorary doctorate on Professor Wilhelm Wirtinger, of the University of Vienna.

Dr. Alexander Ostrowski, of the University of Göttingen, has been awarded a travelling fellowship by the Rockefeller Foundation.

Professor A. S. Eddington, of Cambridge University, has been elected a member of the Athenæum, for "distinguished eminence in science."

Harvard University has awarded from its Milton Fund for Research a sum of money to Professor E. B. Wilson, for expenses connected with investigations in stellar statistics, to be pursued in accordance with biometric methods involving partial correlation.

Professors E. Bompiani, of the University of Bologna, C. Rosati, of the University of Pisa, G. Sannia, of the University of Naples, and G. Vitali, of the University of Padua, have been promoted to full professorships of mathematics.

Professors A. Terracini, of the University of Catania, and F. Tricomi, of the University of Florence, have been transferred to the University of Turin.

The following appointments to associate professorships of mathematics in Italian universities are announced: L. Brusotti and G. Giorgi, at the University of Cagliari; F. Cecioni and G. Marletta, at the University of Catania; G. Sansone, at the University of Florence; E. G. Togliatti, at the University of Genoa; F. Sibirani, at the University of Pavia.

The following have been appointed to instructorships: at the University of Bologna, Dr. A. Agostini, in the history of mathematics, and Dr. E. Bortolotti, in analytic geometry; at the University of Milan, Dr. B. Finzi, in rational mechanics.

Professor Konrad Knopp, of the University of Königsberg, has been appointed professor of mathematics at the University of Tübingen.

At Princeton University, Assistant Professor J. W. Alexander has been promoted to an associate professorship and Dr. T. Y. Thomas has been appointed to an assistant professorship of mathematics.

Professor E. T. Bell, of the University of Washington, has been appointed professor of mathematics at the California Institute of Technology.

Associate Professor J. D. Bond, of the University of Tennessee, has been promoted to a full professorship of mathematics.

Dr. Laura Brant, of Vassar College, has been appointed professor of physics and mathematics at Judson College, Marion, Ala.

Assistant Professor Thomas Buck, of the University of California, has been promoted to an associate professorship of mathematics.

Dr. L. M. Graves has been appointed assistant professor of mathematics at the University of Chicago.

Dr. Jewell C. Hughes, of the University of Arkansas, has been promoted to an assistant professorship of mathematics.

Assistant Professor Glenn James has been promoted to an associate professorship of mathematics at the Southern Branch of the University of California.

Assistant Professor H. E. Jordan, of the University of Kansas, has been promoted to an associate professorship of mathematics.

Associate Professor Louis Lindsey, of Syracuse University, has been promoted to a full professorship of applied mathematics.

Assistant Professor J. C. Morehead has been promoted to an associate professorship at the Carnegie Institute of Technology.

Dr. F. H. Murray, of Dalhousie University, has been promoted to an assistant professorship of mathematics.

Dr. J. R. Musselman, of Johns Hopkins University, has been promoted to an associate professorship of mathematics.

Miss Margaret C. Packer, of Hood College, has been promoted to an assistant professorship of mathematics and physics.

At the University of Michigan, the following appointments to assistant professorships in mathematics are announced: Mr. G. Y. Rainich of Johns Hopkins University, Assistant Professor R. L. Wilder of Ohio State University, and Assistant Professor J. A. Nyswander of Swarthmore College.

Professor W. W. Rankin, of Agnes Scott College, has been appointed professor of mathematics at Duke University.

Assistant Professor L. P. Siceloff, of Columbia University, has been promoted to an associate professorship of mathematics.

Dr. H. A. Simmons, of the University of Pittsburgh, has been appointed assistant professor of mathematics at Northwestern University.

Professor J. D. Tamarkin has been appointed assistant professor of mathematics at Dartmouth College.

Professor J. H. Tanner, head of the department of mathematics at Cornell University, has retired.

Assistant Professor J. S. Taylor, of the University of Pittsburgh, has been promoted to an associate professorship of mathematics.

Assistant Professor J. I. Tracey, of Yale University, has been promoted to an associate professorship of mathematics.

Associate Professor R. M. Winger, of the University of Washington, has been promoted to a full professorship of mathematics.

Dr. Euphemia R. Worthington, of the Southern Branch of the University of California, has been promoted to an assistant professorship of mathematics.

The following appointments to instructorships are announced:

Bryn Mawr College, Dr. Echo D. Pepper;

University of Michigan, Dr. F. W. Kokomoor, Mr. R. H. Marquis, Mr. I. M. Sheffer.

University of Rochester, Miss Rose L. Anderson;

Rutgers University, Dr. C. M. Huber and Mr. A. E. Meder.

Professor E. Czuber, of the Vienna Technical School, died August 22, 1925.

The death is announced of Professor K. Reich, of the Vienna Technical School.

Dr. Paul Roth, of the University of Vienna, died January 17, 1925.

Professor L. G. Gouy, of the University of Lyons, died January 27, 1926, at the age of seventy-two.

Professor S. M. Barton, of the University of the South, died January 5, 1926, at the age of sixty-six. Professor Barton had been a member of the American Mathematical Society since 1892.

Professor R. D. Bohannan, who had been head of the Department of Mathematics at Ohio State University for thirty-nine years, died of apoplexy at Columbus, Ohio, on June 20, 1926, at the age of seventy-one years.

Professor A. A. Heinz, of Tsing Hua College, Peking, China, died suddenly on June 11, at Riverside, California, within a few days after his return to America. He was to have had leave of absence and was to have taught at the University of Washington during the coming academic year.

# NEW PUBLICATIONS

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